Graders mark out of 20. (Anyone who get above 20 raw gets 20.)

1. Find disjunctive and conjunctive normal forms for the following.
   (a) \( a_1 = (x_1 \land x_2) \lor ((\overline{x_3} \lor x_1) \land \overline{x_2}) \lor x_2 \)
   (b) \( a_2 = (x_1 \lor ((x_1 \land x_2) \lor (\overline{x_1} \land \overline{x_2} \land x_3))) \land x_3 \)
   (c) \( a_3 = (x_1x_2 + x_3) \lor (x_1 \rightarrow x_2) \)

   3 Marks

2. Prove that \textit{nand} is \textit{functionally complete}. (To wit: if we let \( p \ast q \) mean \( \neg(p \land q) \) show that the other connectives, \( \land \), \( \lor \), \( \neg \) and \( \rightarrow \) are expressible in terms of \( \ast \).)

   2 Marks

3. Construct truth trees to test the validity of the following claims. If they are invalid, give a counterexample.
   (a) Premisses \( A \rightarrow B \), \( A \lor C \), \( C \rightarrow (A \land B) \).
       Conclusion \( B \).
   (b) Premisses \( p \rightarrow q \), \( p \rightarrow (r \rightarrow q) \), \( \overline{q} + (q \rightarrow p) \)
       Conclusion \( q \rightarrow r \).
   (c) Premisses \( \forall x \forall y \forall z ((Fxz \land Fzy) \rightarrow Fxy) \), \( \forall x \forall y (Fxy \rightarrow Fyx) \)
       Conclusion \( \forall x \forall y (Fxy \rightarrow Fxx) \).
   (d) Hypothesis \( \exists x \forall y Fxy \). Conclusion \( \forall x \exists y Fxy \).

   2,2,2,2 Marks

4. Prove by a \textit{direct} argument that if \( x \) and \( y \) are odd then \( x + y \) is even.

   2 Marks

5. Let \( x \) be an integer. Prove by \textit{contrapositive} that if \( x^2 \) is odd, then \( x \) is odd.

   2 Marks

6. Prove by \textit{contradiction} that if \( x \) is a rational number, and \( y \) is irrational, then \( x + y \) is irrational. (Remember that \( x \) is rational is defined to mean that there exist integers \( p \), \( q \) with \( q \neq 0 \) and \( x = p/q \). To say that a number is irrational means there are \textit{no} such \( p \), \( q \).)

   2 Marks
7. Modifying the proof from lectures, prove that the square root of 3 is irrational. (You may assume that for any prime number \( p \), and for any number \( z \in \mathbb{Z} \), if \( p \) is a factor of \( z^2 \) then \( p \) is a factor of \( z \).)

3 Marks

Tutorial Exercises for the Week 23 July-29 August

1. Compute miniterm and maxterm expressions corresponding to the following
   
   (a) \( a_1 \equiv pqr + (pr + \overline{p} + qr) \).
   
   (b) \( a_2 \equiv \neg(P \lor \neg(P \lor (P \land Q))) \).

2. Show that nor is functionally complete. (Nor is defined as \( p \uparrow q = p + q \))

3. Construct truth trees to test the validity of the following arguments or formulae.
   
   (a) \((r \to (p \to q)) \to ((r \to p) \to (r \to q))\)
   
   (b) \(\forall x(F_x \to Gx) \to (\forall x Fx \to \forall x Gx)\).
   
   (c) Premisses: \(\forall x \forall y \forall z(F_{zy} \land F_{xz}) \to F_{xy}\),
       \(\forall x \forall y \forall z(F_{zy} \land F_{xz}) \to (F_{xy} \lor F_{yx})\)
       \(\forall x \forall y (F_{xy} \to F_{yx})\).

   Conclusion: \(\forall x \forall y ((\exists z(F_x z \land F_{yz}) \to \exists z(F_x z \land F_{yz}))\).

4. Let \( x \) and \( y \) be integers. Prove by contrapositive that if \( xy \) is odd, then \( x \) and \( y \) are odd.

5. Prove by contradiction that if \( x^2 \) is odd, then \( x \) is odd.

6. Is the following argument valid? If Wellington is in New Zealand, then Paris is not in France. But Paris is in France. Therefore Wellington is not in New Zealand.
Some logical equivalences; You may use these in your truth trees as “rules of logic”, as well as Modus Ponens and the and or rules from class. Remember the rules can be used in “either direction” as they are equivalences.

(a) Double Negation \( p \equiv \neg\neg p \).
(b) Double Negation \( p \equiv \neg\neg p \).
(c) De Morgan’s Laws
   (a) \( \neg(p \land q) \equiv \neg p \lor \neg q \)
   (b) \( \neg(p \lor q) \equiv \neg p \land \neg q \)
(d) \( p \rightarrow q \equiv \neg p \lor q \)
(e) \( \neg(p \rightarrow q) \equiv p \land \neg q \)
(f) Commutative Laws
   (a) \( p \land q \equiv q \land p \)
   (b) \( p \lor q \equiv q \lor p \)
(g) Idempotent Laws
   (a) \( p \land p \equiv p \)
   (b) \( p \lor p \equiv p \)
(h) Distributive Laws
   (a) \( p \lor (q \land r) \equiv (p \lor q) \land (p \lor r) \)
   (b) \( p \land (q \lor r) \equiv (p \land q) \lor (p \land r) \)
(i) Associative Laws
   (a) \( p \land (q \land r) \equiv (p \land q) \land r \)
   (b) \( p \lor (q \lor r) \equiv (p \lor q) \lor r \)
(j) Contrapositive \( p \rightarrow q \equiv \neg q \rightarrow \neg p \)
(k) Tautology. If \( T \) is a tautology, then
   \[
p \lor T \equiv T
   \]
   \[
p \land T \equiv p
   \]
(l) Contradiction. If \( F \) is a contradiction, then
   \[
p \lor F \equiv p
   \]
   \[
p \land F \equiv F
   \]

The substitution rule says that we can replace a part of a proposition by a logically equivalent one. More precisely, suppose \( X \equiv Y \). Let \( A \) be a proposition and let \( B \) result by substituting \( Y \) for \( X \) in some place in \( A \). Then \( A \equiv B \).

Using repeated applications of the above laws one can derive that certain statements are logically equivalent or are tautologies. Examples will be given in class.