Bumper Holiday Issue

It is possible to get more than 20 marks for this assignment.

1. How many words of 6 letters can be chosen from a group of 200 letters (a strange alphabet) given that
   (a) Repetition is allowed
   (b) Repetition is not allowed
   (c) Given that the group is broken into two groups of 100 letters, and the first 3 must be chosen from the first group, and the last three from the second; repetition is not allowed.

   Marks 1,1,1

2. An imaginary country is considering possible formats for its number plates. Which of the following formats would produce the largest number of possible number plates? Explain your reasoning.
   (a) Three letters, followed by three digits.
   (b) Seven digits.
   (c) Five letters.

   (Letters are chosen from the English alphabet, and are all upper-case. Digits are chosen from \{0, 1, \ldots, 9\}.)

   Marks 2

3. How many positive integers between 1 and 600 are divisible by either 6 or 10?

   Marks 2

4. Use the Binomial Theorem to find the coefficient of \(x^4\) in \((x + \frac{1}{x})^{10}\).

   Marks 1
5. Let \( A = \{1, 2, 3, 4\} \), and consider the following three relations on \( A \):

\[
\text{id}_A, \quad R = \{(x, y) \mid x \geq y\}, \quad S = \{(x, y) \mid x + y \text{ is odd}\}.
\]

(i) For each relation draw a coordinate diagram, a set diagram, and a directed graph. Marks 3

(ii) List the members of \( \{x \mid x R 3\} \) and \( \{y \mid 2S y\} \). Marks 2

(iii) List the members of \( R^{-1}, RS, R \cap S \). Marks 2

6. Let \( R \) be the equivalence relation on the set \( \{1, 2, \ldots, 7\} \) given by the partition

\[
\{ \{2, 4, 6\}, \{1, 3\}, \{5\}, \{7\} \}.
\]

Draw a graph of \( R \) and describe \( R \) as a set of ordered pairs. Marks 2

7. Prove that the relation

\[
R = \{(x, y) \mid 3 \text{ divides } (x - y)\}
\]

is an equivalence relation on the set \( A = \{0, 1, \ldots, 15\} \), and list all its equivalence classes. Marks 2

8. Let \( R \) be a relation on a set \( A \).

(a) Prove that if \( R \) is symmetric, then \( R \subseteq R^{-1} \).

(b) Prove that if \( R \) is symmetric, then \( RR \) is symmetric.

(c) Prove that if \( R \) is transitive, then \( RR \) is transitive. Marks 3

9. Draw a Hasse diagram of the partial order \( \{(x, y) \mid x \text{ divides } y\} \) on the set \( \{1, 2, \ldots, 15\} \). Marks 2

**Tutorial Exercises for the 5-9 September**

1. Prove that \( \log_3(2) \) is irrational. (Hint: Suppose not and let \( \frac{p}{q} = \log_3(2) \). So \( 3^{\frac{p}{q}} = 2 \).)
2. How many subsets of the set \{a, b, \ldots, j\} contain b?

3. Prove that the sequence \(1 + \frac{1}{2} + \frac{1}{3} + \ldots\) diverges, meaning that at some stage it exceeds any given number thereafter.

   (Hint: Consider \(H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2^n}\). Use induction to prove that \(H_k \geq 1 + \frac{k+1}{2}\).)

4. Let \(A = \{2, 3, 4, 8\}\), and consider the relations

   \[R = \{(x, y) \mid x \text{ divides } y\}, \quad S = \{(x, y) \mid x + y \text{ is even}\}\]

   Draw directed graphs of \(R\) and \(S\), and list the members of \(R^{-1}, S^{-1}, RS, SR, R \cap S, R \cup S, R(R^{-1})\).

5. Think of \(\mathbb{R} \times \mathbb{R}\) as the set of points of the Cartesian plane, and define a relation \(R\) on such points by \(pRq\) iff \(p\) and \(q\) have the same distance from the origin. Explain why \(R\) is an equivalence relation, and describe geometrically the partition of \(\mathbb{R} \times \mathbb{R}\) it induces.

6. Consider the relations on the Cartesian plane \(\mathbb{R}^2\) given by

   \[(u, v)R(x, y) \quad \text{iff} \quad u + y = v + x,\]

   \[(u, v)S(x, y) \quad \text{iff} \quad u^2 + v^2 = x^2 + y^2\]

   In each case, explain why the relation is an equivalence relation, and describe its equivalence classes geometrically.