Assignment 3 Solutions

1. This is done by case analysis. There are many arguments but they run along the following lines. (I would use diagrams...)

Give the 4 corners the names e,w,n,s, (east west etc) Then along the axes, have w2 as the second west one and m for middle, etc. So the centreline is w, w2, m, e2, e. then nw (for north west) etc for the ones on the oblique sides.

Then suppose that w is coloured 1 and n 2. The nw must be 3, so n2 must be 1. Now this implies w2 must be 2 or 3. If w2 2, sw 3, m 3, thus e2 2. But then s2 is 1, so s is 2 (as it is joined to sw and s2). So sw is 3 hence e is 1.

The other cases are similar.

The other representative case is that w is 1 and n is 1. Again if you casde thru things the colouring is forced.

(Incidentally this is used as a “crossing gadget” in math 335 to reduce 3 colouring of arbitrary graphs to 3 colouring of planar graphs... how?)

2. (a) RTP \( \forall n \in \mathbb{N}^+ \) \( (P(n)) \) where \( P(n) \) is the statement

\[
1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.
\]

**Basis** 
\( P(1) : 1^2 = \frac{1(1+1)(2(1)+1)}{6} = \frac{6}{6} \quad \checkmark \)

**Induction Step** 
RTP: \( P(k) \Rightarrow P(k+1) \)

So suppose \( P(k) : 1^1 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6} \)

then RTP \[
1^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}.
\]

\[
\begin{align*}
LHS & = 1^2 + \cdots + (k+1)^2 = (1 + 2 + \cdots + k^2) + (k+1)^2 \\
 & = \frac{1}{6} k(k+1)(2k+1) + (k+1)^2 \quad \text{by } P(k) \\
 & = \frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6} = \frac{(k+1)(k+2)(2(k+1)+1)}{6}
\end{align*}
\]

after some algebra (YOU are not allowed to say this of course, YOU must do the relevant algebra.).

**Marks 2**

(b) RTP \( \forall n \geq 0 \) \( P(n) : 1 + 2 + \cdots + 2^n = 2^{n+1} - 1 \)

**Basis** 
\( P(0) : 1 = 2^{0+1} - 1 = 2 - 1 \quad \checkmark \)
**Induction Step**

**RTP:** For all \( k \), \( P(k) \Rightarrow P(k+1) \)

That is, if \( P(k) : 1 + \cdots + 2^k = 2^{k+1} - 1 \)

then \( 1 + \cdots + 2^{k+1} = 2^{k+2} - 1. \)

So suppose \( 1 + \cdots + 2^k = 2^{k+1} - 1 \),

then RTP: \( 1 + \cdots + 2^{k+1} = 2^{k+2} - 1. \)

Now

\[
LHS = 1 + 2 + \cdots + 2^{k+1} = (1 + 2 + \cdots + 2^k) + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} \quad \text{by Hypothesis} = 2(2^{k+1}) - 1 = 2^{k+2} - 1
\]

as required.

(c) RTP 3 is a factor of \( 7^n + 2 \) for all \( n \).

**Basis** \( n = 0 \). \( 7^0 + 2 = 1 + 2 = 3. \)

Assume that 3 is a factor of \( 7^k + 2 \). RTP 3 is a factor of \( 7^{k+1} + 2 \).

Now, \( 7^{k+1} + 2 = 7(7^k) + 2 = 7(7^k) + 14 - 14 + 2 = 7(7^k + 2) - 12. \) Now 3 is a factor of \( 7^k + 2 \) by hypothesis, and hence of \( 7(7^k + 2) \), and 3 is a factor of 12, and hence 3 is a factor of \( 7(7^k + 2) - 12 \), as required.

Marks 2

3. RTP: \( \sqrt{3} \) is irrational.

Proof by contradiction:

Assume \( \sqrt{3} \) is rational. \((*)\)

Then for some \( p, q \) in \( \mathbb{Z} \), \( q \neq 0 \), we have \( \sqrt{3} = \frac{p}{q} \). By dividing out we can have that \( p, q \) have no common factors. \((1)\)

Now \( 3 = \sqrt{3}^2 = \frac{p^2}{q^2} \).

Hence \( 3q^2 = p^2 \).

Therefore 3 is a factor of \( p^2 \) and hence 3 is a factor of \( p \). \((2)\)

(They are allowed this assumption)

Thus \( p = 3n \) for some \( n \in \mathbb{Z} \).

Hence \( 3q^2 = (3n)^2 = 3^2n^2 \) and hence \( q^2 = 3n^2 \).

Therefore 3 is a factor of \( q^2 \) and hence of \( q \). Marks 2

4. RTP: Any score of 5 or more can be achieved by using only 2-cent and 5-cent stamps.

**Base case:** \( n = 5 \). Use a 5.

Suppose for \( n = k \), and consider \( n = k + 1 \). We need one more. If a 5 has been used, we can replace it by 6 obtained as 3 two’s.
If, for \( k \), we used no 5’s, then the are all 2’s. Since \( n \geq 5 \), \( k \geq 6 \) and hence we can replace 6 by 7 by taking out 3 two’s and replacing with a 5 and a 2.

Marks 2

5. One marks for each

(i) \( \exists x(S_x \land Px) \)

(ii) \( \forall x(Px \rightarrow \neg Sx) \)

(iii) \( \forall x(Px \rightarrow \exists y(Fy \land Wxy)) \)

(iv) \( \forall x((Px \land Sx) \rightarrow \neg \exists y(Fy \land Wxy)) \)

(v) \( \forall x(Fx \rightarrow \exists y(Py \land Fyx)) \)

6. Use induction on \( j \) where \( k = 2^j \) is the number of points. If \( j = 1 \) easy. start with 0 then move to 1.

Assume for \( j \). Take a disc with \( 2^j + 2 \) many points. Somewhere on the circle there must be a 0 followed by a 1. If I remove these two consecutive points \( a, b \), then the remaining disc has only \( 2^j \) many points and I can apply induction to get an ordering \( x_1, \ldots, x_{2^j} \), with the property that the 1’s never exceed the 0’s. In the original list, the pair \( a, b \) will have appeared. For example, \( a \) will have appeared immediately after \( x_i \) and \( b \) immediately before \( x_{i+1} \). If we slot them back in, we know position \( a \) is a zero, so \( x_1, \ldots, x_i, a \) is acceptable as I only increase the 0’s by 1, and by the same token \( x_1, \ldots, x_i, a, b \) is also okay as I have only increased the overall number of zero’s and 1’s by one for each of \( a \) and \( b \). Thus I can slot \( a, b \) into the sequence to get \( x_1, \ldots, x_i, a, b, x_{i+1}, \ldots, x_{2^j} \).

The astute reader will note that there’s a nice recursive algorithm here to find the ordering.

Marks 2