Logical Arguments

Consists of

Premises (Hypotheses)

\[ P_1, \ldots, P_n \]

& a conclusion \( C \).

Valid if \( C \) follows from

\[ (P_1 \land \lnot P_n) \rightarrow C. \]

\[ P_1, \ldots, P_n \]

being true.

Implies \( C \) is true.

Using "logically correct" reasoning.
Many techniques we will look at.

No algorithm to do it.

* in fact we can prove there is no algorithm.

Certain proof techniques.

So far we have seen 2 techniques.

In propositional logic.
(1) Truth tables
(2) "logical equivalences"

"Axioms" = properties of \( \equiv \)

\[
P_1 \equiv P_2
\]

1 C.

Every line followed by a previous line via one of the equivalences (\(+\) a substitution)
Directed Acyclic Graph

\[ \text{A} \rightarrow \text{B} \land (\text{C} \lor \text{D}) \]

\[ (\text{A} \lor (\text{C} \land \text{B})) \land (\text{D} \lor \text{E}) \]

\[ (\text{F} \land \neg \text{E}) \lor (\text{D} \land \text{E}) \]
Every line is a sequence of previous lines.

First formalized by Greeks, Aristotelian syllogisms.
Simple "number" examples.

\[ \mathbb{Z} = \{-\ldots, -2, -1, 0, 1, 2, 3, \ldots\} \]

Say \( x \) is even if
\[ x = 2n \quad \text{some} \quad n \in \mathbb{Z} \]

odd \( x = 2n+1 \quad \text{some} \quad n \in \mathbb{Z} \]

-1 odd

2 even

allow rules of basic arithmetic.
Claim

\[ \begin{array}{c}
\text{if } x \text{ is odd} \\
\text{then } x^2 \text{ is odd} \\
\end{array} \]

Premise: \( x \) is odd.

Hyp: \( x = 2n + 1 \) some \( n \) in \( \mathbb{Z} \).

RTP: required to prove.

\[ x^2 = 2n + 1 \]

\[ x^2 = (2n+1)^2 = (2n)^2 + 4n + 1 \]

\[ = 2 (2n^2 + 2n) + 1 \]

\[ = 2 \square + 1 \]
\[(a+b)^2 = a^2 + 2ab + b^2\]

High school.

\[x \text{ is even}\]

Hyp: \[5x \text{ is even}\]

So \[x = 2n\] some \[n \in \mathbb{Z}\].

So \[5x = 5(2n)\]

So \[5x = 2(5n) = 2\]

Direct proof.
\[ P \in \mathbb{C} \quad \Rightarrow \quad (x, y) \in \mathbb{R} \times \mathbb{R} \]
\[ \text{for } x = 3, y = 0 \text{ odd} \]
\[ = \quad 1 \times 15 = 15 \text{ odd} \]

Conclusion.

Hypothesis

Assume some to be

Assumptions

Remains

There are more
Assume: $x = 2n + 1$ for some $n \in \mathbb{Z}$ and $n \geq 4$.

RTP: $x^3 = 2m + 1$ for some $m \in \mathbb{Z}$.

$x = 2n + 1$.

Therefore: $x^3 = (2n+1)^3$.

$\therefore x^3 = (2n)^3 + 3(2n)^2 + 3(2n) + 1$

$= 8n^3 + 12n^2 + 6n + 1$

$= 2(4n^3 + 6n^2 + 3n) + 1$

$= 2m + 1$

For $m = 4n^3 + 6n^2 + 3n$. 
Proof by contra positive

\[ \neg P \rightarrow \neg \neg P \]

So to prove \( P \rightarrow Q \)

enough to prove \( \neg Q \rightarrow \neg P \)

Assume \( \neg Q \rightarrow \neg P \)

\[ \neg P \]

\( \neg P \) (premiss)
Claim

\[ X^2 \text{ is even} \]

Then

\[ X \text{ is even} \]

\[ P \rightarrow \exists q \]

\[ P \rightarrow q \]

Proof:

Contradiction:

\[ \neg P \rightarrow q \rightarrow \neg P \]

Let suppose

\[ X \text{ is not even} \]

\[ \Rightarrow X \text{ is odd} \]

\[ \Rightarrow X = 2n + 1 \text{ some } n. \]

\[ \therefore x^2 = (2n + 1)^2 \]

\[ = 4n^2 + 4n + 1 \]

\[ = 2(m + 1) \text{ even } \]

\[ \Rightarrow x^2 \text{ odd } \]
If \( x \) is odd, \( y \) is even (not odd).

If \( x \) is even, \( y \) is odd.