Q2. (a) \( Q = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (3,2), (3,3), (4,4), (4,5), (5,4), (5,5)\} \)

(b) \( A \cap B \) omitted

(iii) \( A = \text{badminton} \), \( B = \text{tennis} \)
\[ |A| = 20, \quad |B| = 10 \]
\[ |A \cup B| = |A| + |B| = 30 \]
\[ |A \cap B| = 25 \]

\[ |A \cup B| = |A| + |B| - |A \cap B| \]
\[ = 20 + 10 - 25 \]
\[ = 5 \]
\[ 30 - 2 \times |A \cap B| = 25 \]
\[ |A \cap B| = 2.5 \]
(Probably) Probably, "either" should be dropped from the question, in which case

\[ |A \cup B| = 25 \]
\[ = |A| + |B| - |A \cap B| \]
\[ \Rightarrow |A \cap B| = 5. \]

(c) transitive \( \Rightarrow \) if \((a, b), (b, c) \in R\), then \((a, c) \in R\).

In fact \( R \) is transitive \( \iff R \circ R \subset R \).

(See the lecture notes)

(d) To show \( 6 \mid n^3 + 5n \), it is sufficient to show \( 2 \mid n^3 + 5n \), and \( 3 \mid n^3 + 5n \).

\[ n^3 + 5n = n(n^2 + 5) \]

If \( n \) is even, \( 2 \mid n \) so \( 2 \mid n(n^2 + 5) ? \)

If \( n \) is odd, \( n^2 + 5 \) is even.

So, \( 2 \mid n^3 + 5 \), so \( 2 \mid n(n^2 + 5) \).

If \( 3 \mid n \), then \( 3 \mid n(n^2 + 5) \).
If 3 ∤ n, then n = 3m + 1 or 3m + 2 for some m ∈ ℤ.

Then, n^2 + 5 = 0. If n = 3m + 1,

n^2 + 5 = 9m^2 + 6m + 6, so 3 | n^2 + 5.

If n = 3m + 2,

n^2 + 5 = 9m^2 + 12m + 9, so 3 | n^2 + 5.

In all cases, 3 | n(n^2 + 5).

So, 6 | n(n^2 + 5).

Q3. (a) Omitted

(c) S is symmetric, R is antisymmetric

If x | y, thus x ≤ y, and y | x, thus y ≤ x, then x = y. So R is partial.

\[ R = \{(1, 1), (1, 2), (1, 3), (4, 1), (5, 1), (6, 2), (2, 4), (2, 6), (3, 3), (5, 6), (6, 6)\} \]

Hasse:

```
  4 6
 /|
2 3
```

1 5
(iii) (iv): Omitted

(b) Suppose \( E = \{1, 2, \ldots, n\} \) is a set of size \( n \)

- \# of ways to choose a subset \( S \) of \( E = 2^n \) because for each \( i \in E \), \( i \) is either in \( S \) or not in \( S \).
- So, for each \( i \), \( \# = 2 \) choices.
- So, \# of subsets \( = 2^n \).

(c) Induction: When \( n = 1 \), the only deficient \( 2 \times 2 \) board is \( \square \), itself a minimax.

Suppose it is true for \( n \).

Then, for a deficient \( 2^{n+1} \times 2^{n+1} \) board, divide it into 4 parts:

- The part with deficiency
A, B, C (minus E), D are all deficient \(2^n \times 2^n\) boards, so we can fill them up by trinomials, and E is itself a trinomial.

Q4. (a) (i) \(S = \{x \mid xy = 1\}\)

\[
S = \{x \mid y = \frac{1}{x}\}
\]

When \(x = 0\), \(\frac{1}{0}\) is not defined.

(ii) \(S\) is well-defined for \(\text{dom}(S) = \{x \in \mathbb{R} | x \neq 0\}\)

But \(\frac{1}{x} = 0\) for any \(x\), so, not onto.

(b) Suppose \(g \circ f(x) = g(f(x)) = g(f(x'))\)

\(g(f(x)) = g(f(x'))\),

Since \(g\) is \(1-1\), \(f(x) = f(x')\).

Since \(f\) is \(1-1\), \(x = x'\).

So, whenever \(g \circ f(x) = g \circ f(x')\), \(x = x'\).
(c) $\emptyset 4 \times 4 \times 4 \times 4 \times 4 \times 4 = 4^6$

1-1 function = 0. (domain size > range size)

\[ \text{Q5. (a)} \]

\[ \frac{35}{100} \]

\[ \frac{30}{30} \]

\[ \frac{30}{5} = \gcd \]

\[ 5 = 35 - 1 \cdot 30 \]

\[ 30 = 100 - 2 \cdot 15 \]

\[ 5 = 35 - 1 \cdot (100 - 2 \cdot 35) \]

\[ = 35 - 1 \cdot 100 + 2 \cdot 35 \]

\[ = 3 \cdot 35 - 1 \cdot 100 \]

\[ (b) \]

\[ a = 2 \cdot 3 \cdot 5^2 \quad b = 2^3 \cdot 3^2 \cdot 5 \cdot 11^4 \]

(i) \[ \gcd = 2 \cdot 3 \cdot 5 \quad \text{lcm} = 2^3 \cdot 3^2 \cdot 5 \cdot 11^4 \]

(ii) \[ (3+1)(2+1)(1+1)(4+1) = 120 \]
Q6. (a) (i), (ii), (iii): omitted
(b) (i) $K_5$: $v = 5$, $e = \binom{5}{2} = \frac{5 \cdot 4}{2} = 10$
  $e \geq 3 \cdot v - 6$

  $K_{3,3}$: $v = 3 + 3$, $e = 3 \cdot 3$
  $e \leq 3 \cdot v - 6$

(c) $\exists a, b, c, d, g, i^2$