Historically, mathematicians identified properties of relations which seem important.

Tend to be relations "like" "<" "≤" "="

Let $R \subseteq A \times A$

we say that $R$ is

you must know the definitions
(1) Reflexive

If, for all \( x \in A \),
\( x \mathbin{R} x \).

(2) Graph

\( x \) \( \rightarrow \) anti-
\( \mathcal{R} \) Reflexive

If, for all \( x \in A \),
\( \neg (x \mathbin{R} x) \).

(3) Irreflexive

not reflexive

\( \mathcal{Q} \) \( \mathcal{Q} \)
\( 1 \ 2 \ 3 \)
(IV) Symmetric

\[ \forall x \forall y \in A \quad \text{if } x R y \quad \text{then } y R x \]

asserts existence of arrows if other arrows exist.

\[ \phi \text{ id}_R \phi \text{ on } \{1, 2\} \text{ Symm.} \]
\{1, 2, 3, 4\}

\{ (1, 2), (2, 3), (3, 4), (4, 3) \}

\(3, 2\)

not symm

missing

anti-refl.
(v) **Anti-symmetric**

for all \( x, y \)

if \( x \mathbin{R} y \) and \( x \neq y \)

then \( (y \mathbin{R} x) \)

if \( x \mathbin{R} y \) and \( y \mathbin{R} x \) then \( x = y \). 

\[ \leq \]

\[ \uparrow \]
reflexive anti symmetric

met is not as
Transitive

for all \( x, y, z \):

if \((x R y \text{ and } y R z)\)

then \(x R z\)

"is a descendency"
Ga 5.2

r ✓
anti-r X
180° X

Symm ✓
anti-s X
trans ✓
Notes  
Assume $A \neq \emptyset$

$R$ is

(i) reflexive iff $\text{id}_A \subseteq R$

(ii) symmetric iff $R^{-1} \subseteq R$

(iii) transitive iff $RR \subseteq R$
Equivalence relations are those that seem to "look like =" 

Def: R is an equivalence rel

If R is reflexive symmetric & transitive

Examples:
1. =
2. \( A = \{ \text{people} \} \) &
   \( R = \text{"is the same age as"}\)
3. \( A = \{ \text{lines in } R \} \)
   \( \text{a R b if "a is parallel to b"} \)
Classic one

Congruence mod n

\[ a \equiv b \pmod{n} \text{ on } \mathbb{Z} \]

Means that a & b have

The same remainder when divided by n

\[ a = nq + r \quad , \quad b = nq' + r \]

\[ -2 \equiv 3 \equiv 8 \pmod{5} \]

\[ 103 \equiv 3 \pmod{} \]
\[ \mod 5 \]

\[ [i] = \{ a \in \mathbb{Z} \mid a \equiv i \pmod{5} \} \]

\[ \uparrow \]

equivalence class of \( i \)

congruence

\[ \begin{align*}
[0] & = \{ 0, 5, 10, 15, 20, 25, -5, -10, \ldots \} \\
[1] & = \{ 1, 6, 11, 16, -4, -9, \ldots \} \\
[2] & = \{ 2, 7, 12, 17, -3, -8, \ldots \} \\
[3] & = \{ 3, 8, 13, \ldots \} \\
[4] & = \{ 4, 9, 14, \ldots \}
\end{align*} \]
Properties

5 equivalence classes

every number is in exactly one of them

\[
\begin{array}{cccccc}
\text{6} & \text{7} & \text{8} & \text{9} & \text{10} & \text{11} \\
\end{array}
\]

"Cells"

Partition disjoint collection of sets \( A_1, \ldots, A_k \)

(1) \( A = A_1 \cup A_2 \cup \ldots \cup A_k \)
(2) \( A_i \cap A_j = \emptyset \), \( \forall i \neq j \)
(3) \( A_i \neq \emptyset \), \( \forall i \)
In general, if \( R \) is an equivalence relation, define the \( R \)-equivalence class

\[ [a]_R = \{ b \mid aRb \} \quad a \in A. \]

\( R \) generates a partition

"equivalence rel's = partition"
A = \{1, 2, 3, 4\}

R = \{ (1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2) \}

\Pi_K = \{ 1, 3, 5, 4 \}