Logical Arguments

Premises

$P_1, \ldots, P_n$ \[ \vdash C \]

Conclusion $C$

Valid if $C$ followed from $P_1, \ldots, P_n$ being true by "logically correct reasoning"
Codify various proof techniques (cf. properties $g \equiv$)

Direct proof

\[ P_1 \\
\vdots \\
P_n \]

\{ every line is a consequence of earlier ones \}
Some simple "number"

Examples

\[ \mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \} \]

Integers

Say \( x \) is even if \( x = 2a \) for some integer \( a \).

eg. \( 4, -6 \)

Say \( x \) is odd if \( x = 2b + 1 \) for some \( b \).

eg. \( -1, 3 \)

Allow rules of basic arithmetic
Claim if \( x \) is odd \( P \)

\[ x = 2p + 1 \]

\[ x^2 \] is odd \( c \)

\[ x^2 = 2^p + 1 \]

**Direct proof**

\[ P: x \text{ is odd.} \]

\[ x = 2a + 1 \quad \text{some } a \in \mathbb{Z} \]

\[ x^2 = (2a + 1)^2 = 4a^2 + 4a + 1 \]

\[ = 2(2a^2 + 2a) + 1 \]

of the form \( 2q + 1 \)

\[ x^2 \text{ is odd} \]

\[ \Rightarrow (a + b)^2 = a^2 + 2ab + b^2 \]
If $x$ is even, $5x$ is even.

Premise

$x$ is even.

\[ x = 2a \text{ some } a \text{ at } 4 \]

\[ 5x = 5(2a) = 10a = 2(5a) \]

\[ = 2 \cdot 5 \text{ some 5} \]
Proof by contra positive

"Know" \( (p \Rightarrow q) \equiv (\neg q \Rightarrow \neg p) \)

do prove \( p \Rightarrow q \)

show equivalence

\( \forall r \Rightarrow \neg r \)

ie to show premises

imply the conclusion

assume the conclusion false

show at least one premise is false.
Example

Show if $x^2$ is even then $x$ is even.

Required to prove: $x$ is not even

RTP: if $x$ is not even

\[ x = 2a + 1 \] name $a$

already proved $\Rightarrow x^2$ is odd

\[ \therefore x^2 \text{ not even} \]
Method of contradiction.

already seen in the

ark trees.

to prove \( P \) assume \( \neg P \)

show this leads to

a contradiction.
"Remember"

A number is called rational if it is a "ratio"

\[ \frac{3}{4}, \frac{22}{9} \]

\[ \frac{1}{2}, \quad \forall \, q \in \mathbb{Z}, \, q \neq 0 \]

Something I assume all rational numbers have "lowest terms"

\[ \not\exists \quad \text{no common factors} \]
\[ \frac{6}{9} \times 3 \text{ divides } 6 \]
\[ \frac{2}{3} \]

\( \mathbb{R}^m \text{ (in Euclid)} \)
\( \sqrt{2} \text{ is not rational} \)

\[ \text{ie no } \frac{a}{b} \text{ s.t. } \sqrt{2} = \frac{a}{b} \]
Suppose not:

\[ \sqrt{2} = \frac{p}{q} \text{ some } p, q \neq 0 \]

\[ q \text{ lowest terms.} \]

\[ (\sqrt{2})^2 = \left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} \]

\[ 2 = \frac{p^2}{q^2} \]

\[ p^2 = 2q^2 \]

\[ p^2 \text{ is even} \]

\[ p \text{ is even.} \]

\[ p = 2b \text{ some } b \in \mathbb{Z} \]

\[ (2b)^2 = 2q^2 = 4b^2 \]

\[ q^2 = 2b^2 \]

\[ q^2 \text{ even} \]

\[ q \text{ even} \]

A contradiction as \( p, q \) lowest terms.
Non constructive proof

Prove something true, don't know why.

- Exist 2 irrational numbers a, b such that a^b is irrational

1st try: a = \sqrt{2}, b = \sqrt{2}

if \sqrt{2}^\sqrt{2} is rational done.

not rational try:

(\sqrt{2})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2})^2} = \sqrt{2}^2 = 2