Statements like “All dogs have four legs” or “Some dogs are friendly” are more complex than simple propositions like “Fido has four legs” or “Fido bites”. Words like “all” or “some” or “there exists” are quantifiers and logic needs to deal with them.

For notation, we let $p(x)$ be an expression making an assertion about $x$, where $x$ takes a range of possible values. Here $p(x)$ is called a propositional function or predicate. It becomes a proposition when $x$ is given a particular value.

**Universal Quantifier** $\forall x \ p(x)$ is read “for all $x$, $p(x)$” or “every $x$ satisfies $p(x)$”.

**Existential Quantifiers** $\exists x \ p(x)$ is read “there exists an $x$ such that $p(x)$” or “for some $x$, $p(x)$”.

**Examples**

(1) “There exists a tall person.”
   Let $p(x)$ mean $x$ is a person and $t(x)$ mean that $x$ is tall.
   Then the statement becomes
   $$\exists x (p(x) \land t(x))$$

(2) “All dogs have four legs”
   Here let $d(x)$ denote “$x$ is a dog” and $l(x)$ denote “$x$ has four legs.” Then we have
   $$\forall x (d(x) \rightarrow l(x))$$

(3) “There exist two primes whose sum is prime.”
   Let $p(x)$ denote “$x$ is prime”. We have
   $$\exists x \exists y (p(x) \land p(y) \land p(x + y))$$

(4) “Every even number greater than 3 is the sum of two primes”
   Here $e(n)$ means “$n$ is an even number.”
∀n((e(n) ∧ (n > 3)) → ∃x∃y(p(x) ∧ p(y) ∧ (n = x + y)))

**Negating Quantifiers** The following relations are fundamental. They are also common sense if you think about them.

¬(∃x p(x)) is equivalent to ∀x ¬p(x).

¬(∀x p(x)) is equivalent to ∃x ¬p(x).

**Restricted Quantifiers** Most often when we use quantifiers we have some underlying set of possibilities in mind. For example, when we say “everyone wants a better future for their children” it is implicit that we are quantifying over the set of people with children. This is an example of a bounded quantifier.

∀x(x ∈ A → p)

is abbreviated to

(∀x ∈ A) p

Both mean “every x in A has property p. Also

∃x(x ∈ A ∧ p)

abbreviates to

(∃x ∈ A) p

Both mean “some x in A has property p.”

Note that we have used p above when strictly speaking we should have used p(x). We will continue to do this at times. This is an example of **mathematical laziness**. It is unforgivable, but all mathematicians use sloppy, ambiguous notation from time to time and you the student need to be prepared to deal with it.
Negating restricted quantifiers we have

\[ \neg (\forall x \in A \ p) \equiv (\exists x \in A) \neg p \]

\[ \neg (\exists x \in A \ p) \equiv (\forall x \in A) \neg p \]

**Order of Quantification Matters** Let \( x \) and \( y \) take positive integers 1, 2, 3, \ldots as values then one of the following statements is true and the other false. Work out which is which and why.

\[ \forall x \exists y (y > x) \]
\[ \exists y \forall x (y > x) \]

Examples of simplifying expressions involving quantifiers will be given in the lectures.