Logical Arguments

An argument consists of

1. a list $P_1, P_2, \ldots, P_n$ of statements called the premisses
2. a statement $C$ called the conclusion.

The following are arguments.

$P_1$ If John got an A he passed MATH 161.
$P_2$ John got an A.
$C$ He passed MATH 161.

$P_1$ If John got an A he passed MATH 161.
$P_2$ He passed MATH 161.
$C$ John got an A.

An argument is valid if the conclusion is true whenever the premisses are true. Clearly the first argument above is valid while the second is not.

- In a valid argument the truth of the premisses forces the truth of the conclusion.
- Given a valid argument, there is no guarantee at all that the conclusion is true. Before you can be sure that the conclusion is true you have to know that the premisses are true.

Testing Validity There are a variety of ways to test that an argument is valid. To some extent this is the point of studying logic. One such test for validity follows. It works if none of the premisses involve quantifiers. Construct truth tables for $P_1, P_2, \ldots, P_n$ and $C$. If there is a row with $P_1, P_2, \ldots, P_n$ all 1 and $C$ having 0, the argument is invalid. (This is because we have found a situation in which the premisses are all true, but the conclusion false.) Otherwise the argument is valid. Examples using this method will be given in class.
We have only touched the subject of formal logic. Sadly time forbids us to do more. Now we turn to the most important application of logic for the mathematician.

**Mathematical Proofs**

In mathematics we try very hard to produce valid arguments. When we find one we call it a **proof** and the conclusion we come to with such a valid argument is called a **theorem**. Learning how to recognize and produce valid proofs is a skill that has to be learnt — it takes time and effort. It is extremely helpful to be able to recognise certain **styles** of proofs.

Often proofs are **direct**, that is they argue from the premises to the conclusion by a direct chain of reasoning. But it is also quite common to argue differently.

**Proof by Contrapositive** We already know from logic that

\[ p \rightarrow q \equiv \neg q \rightarrow \neg p \]

This means that if we want to prove a theorem of the form, “if P then Q”, it is equivalent to prove “if not P then not Q”.

**Theorem** Let \( n \) be a natural number. If \( 3n \) is odd, then \( n \) is odd.

*Proof*. We prove the contrapositive; that is, we show that if \( n \) is not odd, then \( 3n \) is not odd.

Assume that \( n \) is not odd. Then \( n \) is even. Thus \( n = 2m \) for some number \( m \). Now \( 3m = 3.2m = 2(3m) \). Thus \( 3m \) is a multiple of 2 and is therefore even and hence not odd. \( \square \)

Another important technique that often gets confused with the contrapositive — even by mathematicians — is **Proof by Contradiction**. Here we assume the conclusion of the theorem is false and show that is leads to something that contradicts something that we know to be true. Therefore the conclusion of the theorem is true.

Sound confusing? It’s not. We use this reasoning in day to day life. How do we prove that Wellington does not have a tropical climate? Let’s assume the opposite that Wellington does have a tropical climate.
Then it would have lots of palm trees, contradicting the fact that it doesn’t. Therefore Wellington does not have a tropical climate.

I know that was a silly example. I’m typing this late at night and can’t think of anything better. But at least you should get the point. Here is a mathematical example.

In geometry we have the theorem “Given two points, there is exactly one straight line that contains them both.” This uses the axiom from geometry “that there is exactly one line between any two points”.

**Theorem** If distinct lines $m, n$ intersect, they do so in exactly one point.

**Proof.** Assume that there are more than two points on the intersection of $m$ and $n$. Say $p$ and $q$ are two of them. Then there is more than one line containing the points $p, q$, namely $m$ and $n$. But now we have contradicted the theorem we stated before. Therefore the theorem holds. □

Something else is illustrated by the above proof. That is, once we have established a theorem in mathematics we can use it in the process of building others. That way we establish a body of knowledge.

**Question** How can you turn a proof by contrapositive into a proof by contradiction?

Many theorems are of the form “if $p$ then $q$” or $p \rightarrow q$. Here is one:

“If a number is divisible by 4, then it is even”

The **converse** of such a theorem is obtained by reversing the direction of the implication. The converse of the above theorem is

“If a number is even, then it is divisible by 4”

- Usually the converse of a theorem is **not** true.
- The converse is quite different from the contrapositive.

Sometimes both a theorem and its converse are true; in which case we state the theorem as an equivalence, ie $p \equiv q$, typically using “if and
only if”. Such a theorem is really two theorems in one and we usually prove it by first proving $p \implies q$ and then proving $q \implies p$.

**Theorem** A number $n$ is even if and only if $2n$ is divisible by 4.

*Proof.* Assume $n$ is even. Then $n = 2m$ for some number $m$. Now $2n = 2 \cdot 2m = 4m$, so $2m$ is divisible by 4. This establishes one direction.

Assume that $2n$ is divisible by 4. Then $2n = 4m$ for some number $m$. Hence $n = 2m$, and $n$ is even. This establishes the other direction. □

- It is also quite common to prove $p \iff q$, by proving $p \implies q$ and then proving $\neg p \implies \neg q$. 