You already know set theory. It would be impossible to learn language without an intuitive understanding of set theory. All we do in mathematics is make these intuitive ideas precise. In other words set theory is not hard; it is nothing more than codified common sense.

A set is a collection of objects; these objects being called the members or elements of the set.

Notations \( x \in A \), for \( x \) is in \( A \). \( x \not\in A \) for \( x \) is not in \( A \). There are two basic ways to describe a set.

List the members eg \( \{p, q, r\} \).

- Order in a set is irrelevant. So \( \{p, q, r\} = \{q, r, p\} = \{q, p, r\} \).
- Repetition does not count. So \( \{p, q, r\} = \{p, r, q, p, r\} \).

We can write an infinite set as a list, so long as it is clear how the list continues. We can write the even numbers as

\[
\{2, 4, 6, 8, \ldots\}
\]

and we can write the integers as

\[
\{\ldots, -2, -1, 0, 1, 2, \ldots\}.
\]

Give a property characterising members In practice this is a much more common way to specify sets, even in everyday language. For example we can talk about the set of brown dogs (ie the set whose members have the property of being both brown and a dog) but it would be impossible to attempt to list the members of this set.

\( \{x : x \text{ is a prime number}\} \) describes the set of prime numbers.

In general we write \( \{x : p(x)\} \) for the set of things with property \( p \).

Some basic sets of numbers
$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the set of integers.

$\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ is the set of positive integers.

$\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\}$. Note that $m, n \in \mathbb{Z}$ means that both $m$ and $n$ are in $\mathbb{Z}$.

$\mathbb{R} = \ldots$ is the set of real numbers. How do we describe real numbers?

The irrational numbers are $\{x : x \in \mathbb{R}, x \notin \mathbb{Q}\}$.

$\mathbb{C} = \{x + y\sqrt{-1} : x, y \in \mathbb{R}\}$ is the set of complex numbers.

Two sets are equal if they have the same members.

**A Fundamental Problem.** Bertrand Russell (1905) discovered a basic problem with specifying sets using the second method. We first have to realise that it is quite possible for a set to be a member of itself. Strange? Well, if we were to make a list mentioning all the lists in Victoria University, the list itself would have to be listed — right? Thus, for a set $A$, we can have $A \in A$ or $A \notin A$.

Call a set normal if it does not contain itself as a member. Let $R = \{S : S$ is a normal set}. Is $R$ normal?

Assume that $R$ is not normal. Then $R \in R$, that is, $R$ is a member of $R$. But members of $R$ are normal sets! Thus $R \notin R$. We now have $R \in R$ and $R \notin R$ — a contradiction.

Assume that $R$ is normal. Then $R \notin R$, that is, $R$ is not a member of $R$. This means that $R$ is not normal, and again we have a contradiction.

We conclude that the set $R$ is itself contradictory. This is Russell’s paradox and it shook the very foundations of mathematics in the early 20th century.

We solve the problem by assuming that all elements of our set belong to some fixed universal set $U$. For a property $p$ we are allowed only to construct $\{x : x \in U$ and $p(x)\}$.

The history of set theory and the resolution of Russell’s paradox and other similar ones is one of the most fascinating chapters of the history
of mathematics. But that is another story and we must get back to basics and develop an

Algebra of Sets.

Subsets $A$ is a **subset** of $B$ if every member of $A$ is a member of $B$. Notation $A \subseteq B$.

- $A \subseteq B$ if and only if $\forall x (x \in A \rightarrow x \in B)$.
- If $A$ is *not* a subset of $B$ we write $A \nsubseteq B$.
- If $A \subseteq B$, then $B$ is a **superset** of $A$, and we can also write $B \supseteq A$.

**Theorem 5.1.** For sets $A, B, C$.

(a) $A \subseteq A$
(b) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
(c) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

*Proof.* We prove (b). The others are exercises. Assume $x \in A$. Then, since $A \subseteq B$, we have $x \in B$. But then since $B \subseteq C$, we also have $x \in C$. Thus, if $x \in A$, then $x \in C$, so $A \subseteq C$. $\square$

If $A \subseteq B$ and $A \neq B$, then $A$ is a **proper** subset of $B$.

The **empty set** is the unique set that has no members. Notation $\emptyset$.

**Theorem 5.2.** For any set $A$, $\emptyset \subseteq A$.

*Proof.* Assume $\emptyset \nsubseteq A$. Then there is an element $x$ such that $x \in \emptyset$, but $x \notin A$. But this contradicts the definition of the empty set. $\square$

Set Operations

**Intersection** $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

**Theorem 5.3.**
(a) $A \cap B \subseteq A, B$.
(b) $A \subseteq B$ if and only if $A \cap B = A$.
(c) $C \subseteq A \cap B$ if and only if $C \subseteq A$ and $C \subseteq B$.

Note that $A \cap B \subseteq A, B$ means that $A \cap B$ is a subset of both $A$ and $B$. In a similar way, $a, b \in C$ means that both $a$ and $b$ are in $C$. We’ve used this notation before. Whereabouts?
Union $A \cup B = \{ x : x \in A \text{ or } x \in B \}$.

**Theorem 5.4.**  
(a) $A, B \subseteq (A \cup B)$.  
(b) $A \subseteq B$ if and only if $A \cup B = B$.  
(c) $A \cup B \subseteq C$ if and only if $A \subseteq C$ and $B \subseteq C$.

Set Difference $A - B = \{ x : x \in A \text{ and } x \notin B \}$. Sometimes denoted $A \setminus B$.

**Theorem 5.5.**  
(a) $A - \emptyset = A$.  
(b) $A - B = \emptyset$ if and only if $A \subseteq B$.  
(c) $A - B = A$ if and only if $A \cap B = \emptyset$.

Complement In a situation where there is a clear universal set $U$ (which will depend on context) we can define the complement of a set $-A$ to be $U - A$, that is, $-A = \{ x \in U : x \notin A \}$.

Symmetric Difference $A \Delta B = (A - B) \cup (B - A)$.  

**Theorem 5.6.** $A \Delta B = (A \cup B) - (A \cap B)$.

Now for a bunch of properties of set operations.

**Theorem 5.7.** The following properties hold.

1. **Commutative Laws**  
   (a) $A \cap B = B \cap A$.  
   (b) $A \cup B = B \cup A$.

2. **Associative Laws**  
   (a) $(A \cap B) \cap C = A \cap (B \cap C)$.  
   (b) $(A \cup B) \cup C = A \cup (B \cup C)$.

3. **Distributive Laws**  
   (a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$  
   (b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

4. **Identity Law**  
   (a) $A \cup \emptyset = A$  
   (b) $A \cap \emptyset = \emptyset$.

5. **Idempotence**  
   (a) $A \cap A = A$  
   (b) $A \cup A = A$.

6. **Absorption**  
   (a) $A \cap (A \cup B) = A$  
   (b) $A \cup (A \cap B) = A$.

7. **Complementation**
(a) $\neg(-A) = A$
(b) $\neg\emptyset = U$, $\neg\emptyset = \emptyset$
(c) If $A \subseteq B$, then $(-B) \subseteq (-A)$.

(8) De Morgan’s Laws
(a) $\neg(A \cup B) = (-A) \cap (-B)$
(b) $\neg(A \cap B) = (-A) \cup (-B)$

- You should spend time thinking about the relationship between these properties for sets and the analogous properties for logical connectives.
- All the above properties require proof. It is an excellent exercise for you to prove the lot of them.