Victoria University of Wellington
School of Mathematics and Statistics
Te Kura Mātai Tatauranga

Tutorial week 6: Partial Differential Equations — PDEs

Separation of Variables and Fourier series

Set: Thursday 28 March 2024 — Due: Thursday 18 April 2024 (Week 6)

Read the notes and slides —
all this material will be discussed by the end of week 6.

Tutorial exercises — Week 6

1. Attempt to solve the following PDEs using separation of variables.
   If it is possible, determine the resulting ODEs.

   (a) $tu_{xx} + xu_t = 0$ (b) $u_{xx} + u_{tt} + xu = 0$

   Solution:

   (a): Write $u(x,t) = X(x)T(t)$ then

   $$tX''T + xXT' = 0$$

   Divide by $xtXT$ then

   $$\frac{X''}{xX} = -\frac{T'}{tT} = K$$

   So

   $$X'' = KxX; \qquad T' = -KtT.$$

   (b): Write $u(x,t) = X(x)T(t)$ then

   $$X''T + XT'' + xXT = 0$$

   Divide by XT then

   $$\frac{X''}{X} + \frac{T''}{T} + x = 0$$

   So

   $$\frac{X''}{X} + x = -\frac{T''}{T} = K$$

   That is

   $$X'' + xX = KX; \qquad T'' = -KT.$$

2. Using separation of variables, find a solution to Laplace's equation $u_{xx}+u_{yy}=0$ on the rectangle $0<x<a$, $0<y<b$ with Dirichlet boundary conditions:

$$\begin{aligned} u(0,y)&=f(y), & u(a,y)&=g(y), & 0&<y<b;\\ u(x,0)&=h(x), & u(x,b)&=j(x), & 0&\le x\le a. \end{aligned}$$

Note all *four* edges are non-zero.
Hint: Consider adding the solutions to 4 simpler problems.

Solution:

Consider these four simpler problems $(u_i)_{xx}+(u_i)_{yy}=0$ with boundary conditions:

(a)

$$\begin{aligned} u_1(0,y)&=f(y), & u_1(a,y)&=0, & 0&<y<b;\\ u_1(x,0)&=0, & u_1(x,b)&=0, & 0&\le x\le a. \end{aligned}$$

(b)

$$\begin{aligned} u_2(0,y)&=0, & u_2(a,y)&=g(y), & 0&<y<b;\\ u_2(x,0)&=0, & u_2(x,b)&=0, & 0&\le x\le a. \end{aligned}$$

(c)

$$\begin{aligned} u_3(0,y)&=0, & u_3(a,y)&=0, & 0&<y<b;\\ u_3(x,0)&=h(x), & u_3(x,b)&=0, & 0&\le x\le a. \end{aligned}$$

(d)

$$\begin{aligned} u_4(0,y)&=0, & u_4(a,y)&=0, & 0&<y<b;\\ u_4(x,0)&=0, & u_4(x,b)&=j(x), & 0&\le x\le a. \end{aligned}$$

Each of these 4 sub-problems has only one non-zero edge, and is of the form we explicitly discussed before the break...

Then consider

$$u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y).$$

It satisfies the right PDE, and the appropriate boundary conditions...

3. Using separation of variables, find the solution to Laplace's equation in the semi-infinite strip $0<x<a$, $y>0$ with boundary conditions

$$\begin{aligned} u(0,y)&=0, & y&>0;\\ u(a,y)&=0, & y&>0;\\ u(x,0)&=f(x), & 0&\le x\le a;\\ \lim_{y\to\infty}u(x,y)&=0 & 0&<x<a. \end{aligned}$$

Solution:

First try to separate variables:

$$U(x,y) = X(x)Y(y).$$

Then Laplace's equation becomes

$$X''(x)Y(y) + X(x)Y''(y) = 0$$

Thence, dividing by $X(x)Y(y)$ we have

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = 0$$

Thence we have a separation constant

$$\frac{X''(x)}{X(x)} = -k; \qquad \frac{Y''(y)}{Y(y)} = +k.$$

We have 3 possibilities:

- $k < 0$:
  Then $k = -b^2$ and $X''(x) = b^2 X(x)$ so $X(x) = A\cosh(bx) + B\sinh(bx)$;
  but then the boundary conditions in x imply $X(0) = 0 = X(b)$,
  which in turn implies $X(x) \equiv 0$,
  which is uninteresting.
- $k = 0$:
  Then $X''(x) = 0$ so $X(x) = A + Bx$;
  but then the boundary conditions in x imply $X(0) = 0 = X(b)$,
  which in turn implies $X(x) \equiv 0$,
  which is uninteresting.
- $k > 0$:
  Then $k = +b^2$ and $X''(x) = -b^2 X(x)$ so $X(x) = A\cos(bx) + B\sin(bx)$;
  but then the boundary conditions in x imply $X(0) = 0 = X(b)$,
  which in turn implies $A = 0$ and $\sin(ba) = 0$,
  so $b = n\pi/a$ and $X(x) = B\sin(n\pi x/a)$.
  But now $Y''(y) = +b^2 Y(y)$ with $b > 0$, so $Y(y) = C\exp(by) + D\exp(-by)$;
  but then the asymptotic boundary condition in y implies $Y(\infty) = 0$,
  which in turn implies $C = 0$.

  At this stage we have

$$U(x,y) = X(x)Y(y) = B\sin(n\pi x/a)\; D\exp(-n\pi y/a).$$

Invoking linear superposition

$$U(x,y) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)\exp(-n\pi y/a).$$

This satisfies Laplace's equation and the *three* homogeneous boundary conditions.

The only remaining condition is $U(x,0) = f(x)$ which implies

$$f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a).$$

This in principle determines the E_n and we are done.

4. Prove that the Fourier coefficients satisfy:

$$|A_0| \leq \frac{1}{2L} \int_{-L}^{+L} |f(x)|\, dx; \qquad |A_{n>0}| \leq \frac{1}{L} \int_{-L}^{+L} |f(x)|\, dx;$$

$$B_0 = 0; \qquad |B_{n>0}| \leq \frac{1}{L} \int_{-L}^{+L} |f(x)|\, dx.$$

(Much stronger results are actually known.)

Hint: Remember:

$$A_0 = \frac{1}{2L} \int_{-L}^{+L} f(x)\, dx; \qquad A_{n>0} = \frac{1}{L} \int_{-L}^{+L} f(x)\; \cos(n\pi x/L)\, dx;$$

$$B_0 = 0; \qquad B_{n>0} = \frac{1}{L} \int_{-L}^{+L} f(x)\; \sin(n\pi x/L)\, dx.$$

Solution:

This is merely an application of the standard inequality

$$\left| \int_a^b h(x) dx \right| \leq \int_a^b |h(x)| dx,$$

combined with

$$|f(x)\; \cos(n\pi x/L)| \leq |f(x)|\, |\cos(n\pi x/L)| \leq |f(x)|,$$

and

$$|f(x)\; \sin(n\pi x/L)| \leq |f(x)|\, |\sin(n\pi x/L)| \leq |f(x)|.$$

5. Consider the finite sum:

$$S_M(x) = \frac{4}{\pi} \left\{ \sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots + \frac{\sin([2M+1]\pi x)}{2M+1} \right\},$$

which we saw is of interest in analyzing the Gibbs phenomenon for step functions.

(a) Show that:

$$S_M(x) = 4 \int_0^x \{\cos(\pi u) + \cos(3\pi u) + \cos(5\pi u) + \cdots + \cos([2M+1]\pi u)\}\, du.$$

Solution:

Note that

$$\int_0^x \cos(n\pi u) du = \left. \frac{\sin(n\pi u)}{n\pi} \right|_0^x = \frac{\sin(n\pi x)}{n\pi},$$

and sum from $n = 1$ to $n = 2M + 1$.

(b) Show that:

$$\cos(\pi u) + \cos(3\pi u) + \cos(5\pi u) + \cdots + \cos([2M+1]\pi u) = \frac{\sin([2M+2]\pi u)}{2\sin(\pi u)}$$

Hint: This is "merely" a trig identity.
Hint: Use $e^{i\theta} = \cos\theta + i\sin\theta$, and the well-known series

$$1 + x + x^2 + \cdots + x^m = (1 - x^{m+1})/(1-x).$$

Solution:

Note

$$\cos(n\pi x) = \frac{e^{in\pi x} + e^{-in\pi x}}{2} = \frac{1}{2}\left\{(e^{i\pi x})^n + (e^{-i\pi x})^n\right\}$$

Then

$$\begin{aligned}
&\cos(\pi u) + \cos(3\pi u) + \cos(5\pi u) + \cdots + \cos([2M+1]\pi u) \\
&= \frac{1}{2}\left\{\sum_{n=1,3,5,\ldots 2M+1} (e^{i\pi u})^n + \sum_{n=1,3,5,\ldots 2M+1} (e^{-i\pi u})^n\right\} \\
&= \frac{1}{2}\left\{e^{i\pi u}\sum_{m=0}^{M}(e^{i\pi 2u})^m + e^{-i\pi u}\sum_{m=0}^{M}(e^{-i\pi 2u})^m\right\} \\
&= \frac{1}{2}\left\{e^{i\pi u}\frac{1 - e^{i\pi[M+1]2u}}{1 - e^{i\pi 2u}} + e^{-i\pi u}\frac{1 - e^{-i\pi[M+1]2u}}{1 - e^{-i\pi 2u}}\right\} \\
&= \frac{1}{2}\left\{\frac{1 - e^{i\pi[M+1]2u}}{e^{-i\pi u} - e^{i\pi u}} + \frac{1 - e^{-i\pi[M+1]2u}}{e^{i\pi u} - e^{-i\pi u}}\right\} \\
&= \frac{1}{2}\left\{\frac{e^{i\pi[M+1]2u} - e^{-i\pi[M+1]2u}}{e^{i\pi u} - e^{-i\pi u}}\right\} \\
&= \frac{\sin([2M+2]\pi u)}{2\sin(\pi u)}.
\end{aligned}$$

(c) Show that:

$$S_M(x) = 2\int_0^x \frac{\sin([2M+2]\pi u)}{\sin(\pi u)}du.$$

Solution:

Given the above, this step is now trivial.

(d) Show that:

$$S_M\left(\frac{x}{2M+2}\right) = 2\int_0^x \frac{\sin(\pi u)}{\sin(\pi u/[2M+2])}\frac{du}{2M+2}.$$

Solution:

Given the above, this step is now *almost* trivial.

Note from part (c)

$$S_M\left(\frac{x}{2M+2}\right) = 2\int_0^{\frac{x}{2M+2}} \frac{\sin([2M+2]\pi u)}{\sin(\pi u)}du.$$

Then simply change variables: $u_{new} = (2M+2)u_{old}$.

(e) Show that:

$$\lim_{M\to\infty} S_M\left(\frac{x}{2M+2}\right) = \frac{2}{\pi}\int_0^x \frac{\sin(\pi u)}{u} du = \frac{2}{\pi}\int_0^{\pi x} \frac{\sin(u)}{u} du = \frac{2}{\pi}\,\mathrm{Si}(\pi x).$$

Solution:

Almost trivial.

From the above

$$\lim_{M\to\infty} S_M\left(\frac{x}{2M+2}\right) = 2\lim_{M\to\infty}\int_0^x \frac{\sin(\pi u)}{\sin(\pi u/[2M+2])}\frac{du}{2M+2}.$$

But $\lim_{a\to 0}\sin(ax)/a = x$, so

$$\lim_{M\to\infty} S_M\left(\frac{x}{2M+2}\right) = \frac{2}{\pi}\int_0^x \frac{\sin(\pi u)}{u} du = \frac{2}{\pi}\int_0^{\pi x} \frac{\sin(u)}{u} du = \frac{2}{\pi}\,\mathrm{Si}(\pi x)$$

as asserted.

This is another way of getting to the key result for the (step-function) Gibbs phenomenon. (It is very closely related, *but not identical to*, question 5 of the assignment.)
