VICTORIA UNIVERSITY OF WELLINGTON
SCHOOL OF MATHEMATICS AND STATISTICS
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MATH 301 | DIFFERENTIAL EQUATIONS | 2024

Assignment 5: Partial Differential Equations — PDEs

Set: Monday 4 March 2024 — Due: Friday 15 March 2019 at 23:59

Note 60 points total; 10 points each question.

1. **Classification questions (easy):** **(1 point each; 10 points total)**

   Determine the order of the following PDEs for a function $U(x,y)$, $u(x,y)$, $\psi(x,t)$, or $\Psi(x,y,z)$.
   Decide if they are linear or not, and if linear, whether or not they are homogeneous.
   If nonlinear, decide whether or not they are quasi-linear.

   (a) $aU_{xx} + bU_{yy} = 0$, where $a, b \in \mathbb{R}$ are non-zero.
   Solution: 2^{nd} order; linear; homogeneous; (automatically quasi-linear). **QED**

   (b) $xU_x + yU_y = 0$, where $x, y \in \mathbb{R}$ are non-zero.
   Solution: 1^{st} order; linear; homogeneous; (automatically quasi-linear). **QED**

   (c) $aUU_{xx} + bU_xU_{yy} = 0$, where $a, b \in \mathbb{R}$ non-zero.
   Solution: 2^{nd} order; non-linear; (homogeneity is meaningless); quasi-linear. **QED**

   (d) $\dfrac{\partial^3 U}{\partial^2 x\, \partial y} - \dfrac{\partial U}{\partial y} = x^2 + y^2$
   Solution: 3^{rd} order; linear; non-homogeneous; (automatically quasi-linear). **QED**

   (e) $x^2 U_{yy} - yU_x = U$.
   Solution: 2^{nd} order; linear; homogeneous; (automatically quasi-linear). **QED**

   (f) $x^2 U_{yyyy} - yU_x = U^2$.
   Solution: 4^{nd} order; non-linear; (homogeneity is meaningless); quasi-linear. **QED**

   (g) $-i\partial_t \psi = \frac{1}{2m}\nabla^2\psi + V(x)\psi$.
   Solution: 2^{nd} order; linear; homogeneous; (automatically quasi-linear). **QED**

   (h) $u_{xx}u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y)$.
   Solution: 2^{nd} order; non-linear; (homogeneity is meaningless); not quasi-linear. **QED**

   (i) $U_{xx} + y\, U_{yy} = 0$.
   Solution: 2^{nd} order; linear; homogeneous; (automatically quasi-linear). **QED**

   (j) $(\nabla^2)^2\, \Psi := \left[\partial_x^2 + \partial_y^2 + \partial_z^2\right]^2 \Psi = 0$.
   Solution: 4^{th} order; linear; homogeneous; (automatically quasi-linear). **QED**

2. Find general solutions $U(x,y)$ to the following PDEs (straightforward): **(10 points total)**

(a) $\dfrac{\partial U}{\partial x} = x^2 - y^2.$ **(3 points)**

Solution: Integrate with respect to x, keeping y fixed:

$$U = \int (x^2 - y^2)\,dx = \frac{1}{3}x^3 - xy^2 + F(y),$$

where F is an arbitrary function. **QED**

(b) $\dfrac{\partial U}{\partial x} - \dfrac{\partial U}{\partial y} = e^x e^{-y}.$ [Make an appropriate change of variable.] **(4 points)**

Solution: Try a change of variables $\xi = x - y$, $\eta = x + y$. Then

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}$$

$$\frac{\partial U}{\partial y} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{\partial U}{\partial \xi} + \frac{\partial U}{\partial \eta}$$

$$\Longrightarrow \qquad \frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 2\frac{\partial U}{\partial \xi}$$

Since $e^x e^{-y} = e^{x-y} = e^{\xi}$, the PDE becomes $\dfrac{\partial U}{\partial \xi} = \frac{1}{2}e^{\xi}$.

Thence $U(\xi,\eta) = \int \frac{1}{2}e^{\xi} d\xi + F(\eta) = \frac{1}{2}e^{\xi} + F(\eta)$.

Thence $U(x,y) = \frac{1}{2}e^{x-y} + F(x+y)$. **QED**

(c) $U_{xx} = \sin y$. **(3 points)**

Solution: Integrating twice:

$$U_x = x\sin y + F(y) \qquad \Longrightarrow \qquad U = \frac{1}{2}x^2 \sin y + xF(y) + G(y),$$

where F and G are arbitrary functions. **QED**

3. Find general solutions $U(x,y)$ to the following PDEs (some thinking required): **(10 points total)**

(a) $aU_x + bU_y = 0$. **(2 points)**

Solution: The PDE says $(a,b) \perp (U_x, U_y)$, these vectors are perpendicular.
Therefore $(b,-a) \parallel (U_x, U_y)$, these vectors are parallel.
Therefore

$$U(x,y) = f(bx - ay).$$

(you could also try finding an appropriate change of variables). **QED**

(b) $U_x\, g_y(x,y) - U_y\, g_x(x,y) = 0.$ (Treat $g(x,y)$ as given.) **(2 points)**

Solution: The PDE says $(g_y, -g_x) \perp (U_x, U_y)$, these vectors are perpendicular.
Therefore $(g_x, g_y) \parallel (U_x, U_y)$, these vectors are parallel.
Therefore

$$U(x,y) = f(g(x,y)).$$

Here $g(x,y)$ is given, and $f(g)$ is an arbitrary function. **QED**

(c) $U_{xxyy} = 0.$ **(3 points)**

Solution: Integrating step by step

- $U_{xxy} = f(x).$
- $U_{xx} = yf(x) + g(x).$
- $U_x = y \int f(x)dx + \int g(x)dx + h(y).$
- $U = y \int\int f(x)dxdx + \int\int g(x)dxdx + xh(y) + j(y).$
- Relabelling the arbitrary functions, we finally have:

$$U = yF(x) + G(x) + xH(y) + J(y).$$

- You can always check by differentiating. **QED**

(d) $U_{xx} = y\, U_x + xy.$ **(3 points)**

Solution: Keeping y fixed, you can view this as a first-order linear ODE in U_x.
That is

$$\partial_x(U_x) - y(U_x) = xy.$$

Because it is a first-order linear ODE in U_x, you know it is solvable.
Look for an integrating factor:

$$e^{xy}\, \partial_x \left[e^{-xy}\, U_x\right] = xy$$

Therefore:

$$\partial_x \left[e^{-xy}\, U_x\right] = xy\; e^{-xy}$$

Integrate:

$$e^{-xy}\, U_x = \int xy\; e^{-xy} dx + F(y)$$

Rearrange:

$$U_x = e^{xy} \int xy\; e^{-xy} dx + e^{xy}\; F(y)$$

Integrate:

$$U = \int \left\{ e^{xy} \int xy\; e^{-xy} dx \right\} dx + \int \left\{ e^{xy}\; F(y) \right\} dx + G(y)$$

Rearrange:

$$U = y \int \left\{ e^{xy} \int x\; e^{-xy} dx \right\} dx + F(y) \int \left\{ e^{xy}\; \right\} dx + G(y)$$

Now do as much as you can of the integrals...

$$U = y \int \left\{ e^{xy} \int \left(-\partial_y e^{-xy} \right) dx \right\} dx + F(y) \left\{ \frac{e^{xy}}{y} \right\} + G(y)$$

$$U = -y \int \left\{ e^{xy}\, \partial_y \int e^{-xy} dx \right\} dx + e^{xy} \left\{ \frac{F(y)}{y} \right\} + G(y)$$

$$U = -y \int \left\{ e^{xy}\, \partial_y \left[-\frac{e^{-xy}}{y} \right] \right\} dx + e^{xy} \left\{ \frac{F(y)}{y} \right\} + G(y)$$

$$U = -y \int \left\{ e^{xy} \left[e^{-xy} \left(\frac{x}{y} + \frac{1}{y^2} \right) \right] \right\} dx + e^{xy} \left\{ \frac{F(y)}{y} \right\} + G(y)$$

$$U = -y \int \left\{ \frac{x}{y} + \frac{1}{y^2} \right\} dx + e^{xy} \left\{ \frac{F(y)}{y} \right\} + G(y)$$

$$U = -y \left\{ \frac{1}{2} \frac{x^2}{y} + \frac{x}{y^2} \right\} + e^{xy}\, \tilde{F}(y) + G(y)$$

Finally we have

$$U = -\frac{x^2}{2} + \frac{x}{y} + e^{xy}\, \tilde{F}(y) + G(y)$$

(These integrals could have been done in any of a number of different ways.)
Check:

$$U_x = -x + \frac{1}{y} + y e^{xy} \tilde{F}(y)$$

$$U_{xx} = -1 + y^2 e^{xy} \tilde{F}(y)$$

$$U_{xx} - yU_y = -1 + y^2 e^{xy} \tilde{F}(y) - \left[-xy + y^2 e^{xy} \tilde{F}(y) \right] = xy$$

OK, it solves the PDE and it has two arbitrary functions — we had the right answer.
Alternative technique:
Starting from the PDE $U_{xx} = y\, U_x + xy$ simply integrate in the x direction:

$$U_x = yU + \frac{x^2 y}{2} + F(y).$$

This is a first-order linear DE in the x derivatives... Apply the same logic as above...

$$U_x - yU = \frac{x^2 y}{2} + F(y).$$

$$e^{xy} \partial_x (e^{-xy}\, U) = \frac{x^2 y}{2} + F(y).$$

Etcetera...

QED

4. Eliminate the arbitrary functions from the following and so obtain partial differential equations of which they are the general solution (straightforward): **(10 points total)**

(a) $v = g(x^2 + y^2)$. **(2 points)**

Solution:

One free function, look for a first-order PDE.

$$v_x = 2xg'(x^2 + y^2); \qquad v_y = 2yg'(x^2 + y^2) \quad \Longrightarrow \quad yv_x - xv_y = 0.$$

(b) $v = f(x^2 - y^2)$. **(2 points)**

Solution:

One free function, look for a first-order PDE.

$$v_x = 2xf'(x^2 - y^2); \qquad v_y = -2yf'(x^2 - y^2) \quad \Longrightarrow \quad yv_x + xv_y = 0.$$

(c) $v = f(x^2 - y^2) + g(x^2 + y^2)$. **(3 points)**

Solution:

Two free functions, look for a second-order PDE.

$$\begin{aligned} v_x &= 2xf'(x^2 - y^2) + 2xg'(x^2 + y^2); \\ v_y &= -2yf'(x^2 - y^2) + 2yg'(x^2 + y^2) \\ v_{xx} &= 2f'(x^2 - y^2) + 2g'(x^2 + y^2) + 4x^2 f''(x^2 - y^2) + 4x^2 g''(x^2 + y^2) \\ v_{yy} &= -2f'(x^2 - y^2) + 2g'(x^2 + y^2) + 4y^2 f''(x^2 - y^2) + 4y^2 g''(x^2 + y^2) \end{aligned}$$

Then:

$$y^2 v_{xx} - x^2 v_{yy} = 4(y^2 + x^2)f'(x^2 - y^2) + 4(y^2 - x^2)g'(x^2 + y^2)$$

$$yv_x - xv_y = 4xyf'(x^2 - y^2); \qquad yv_x + xv_y = 4xyg'(x^2 + y^2)$$

So:

$$y^2 v_{xx} - x^2 v_{yy} - (y^2/x)v_x + (x^2/y)v_y = 0.$$

Equivalently:

$$xy^3 v_{xx} - x^3 y\, v_{yy} - y^2\, v_x + x^2\, v_y = 0.$$

(d) $v = h(2x - y) - g(2x + y)$. **(3 points)**

Solution:

Two free functions, look for a second-order PDE.

$$v_x = 2h'(2x - y) - 2g'(2x + y); \qquad v_y = -h'(2x - y) - g'(2x + y)$$

$$v_{xx} = 4h''(2x - y) - 4g''(2x + y); \qquad v_{yy} = h''(2x - y) - g''(2x + y)$$

$$v_{xx} = 4v_{yy}.$$

5. **Euler equation: Elliptic/Parabolic/Hyperbolic** (10 points)

Determine the Euler type (i.e. elliptic, hyperbolic or parabolic) of each of the following PDEs; and obtain the general solution in each case: **(10 points total)**

a. $3U_{xx} + 4U_{xy} - U_{yy} = 0.$ **(2 points)**

Solution:

Discriminant: $h^2 - ab = 2^2 - (3)(-1) = 7.$ Hyperbolic.
Alternative 1 — Determinant: $ab - h^2 = 3(-1) - 2^2 = -7.$ Hyperbolic.
Alternative 2 — Determinant: $\det \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \det \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix} = -7.$ Hyperbolic.

Quadratic: $a + 2hz + bz^2 = 0 = 3 + 4z - z^2 \quad \Longrightarrow$
$z = (-h \pm \sqrt{h^2 - ab})/b = 2 \pm \sqrt{7}.$
Apply algorithm: $U(x, y) = F(x + [2 + \sqrt{7}]y) + G(x + [2 - \sqrt{7}]y).$ **QED**.

b. $U_{xx} - 2U_{xy} + U_{yy} = 0.$ **(2 points)**

Solution:

Discriminant: $h^2 - ab = 1^2 - (1)(1) = 0.$ Parabolic.
Alternative 1— Determinant: $ab - h^2 = (1)(1) - 1^2 = 0.$ Parabolic.
Alternative 2 — Determinant: $\det \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0.$ Parabolic.

Quadratic: $a + 2hz + bz^2 = 0 = 1 + 2z - z^2 \quad \Longrightarrow$
$z = (-h \pm \sqrt{h^2 - ab})/b = 1 \pm 0.$
Apply algorithm: $U(x, y) = (x + cy)\ F(x + y) + G(x + y); \quad (c \neq 1).$
Without loss of generality: $U(x, y) = (x - y)\ F(x + y) + G(x + y).$ **QED**.

c. $4U_{xx} + U_{yy} = 0.$ **(2 points)**

Solution:

Discriminant: $h^2 - ab = 0^2 - (4)(1) = -4.$ Elliptic.
Alternative 1 — Determinant: $ab - h^2 = (4)(1) - 0^2 = 4.$ Elliptic.
Alternative 2 — Determinant: $\det \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \det \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} = 4.$ Elliptic.

Quadratic: $a + 2hz + bz^2 = 0 = 4 + 0z - z^2 \quad \Longrightarrow$
$z = -h \pm \sqrt{h^2 - ab} = \pm 2i.$
Apply algorithm: $U(x, y) = F(x + 2iy) + G(x - 2iy).$ **QED**.

d. $U_{xx} + 4U_{xy} + 4U_{yy} = 0.$ **(2 points)**

Solution:

Discriminant: $h^2 - ab = 2^2 - (1)(4) = 0.$ Parabolic.
Alternative 1 — Determinant: $ab - h^2 = (1)(4)) - 2^2 = 0.$ Parabolic.
Alternative 2 — Determinant: $\det \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = 0.$ Pararbolic.

Quadratic: $a + 2hz + bz^2 = 0 = 1 + 4z + 4z^2 \quad \Longrightarrow$
$z = (-h \pm \sqrt{h^2 - ab})/b = -\frac{1}{2} \pm 0.$
Apply algorithm: $U(x, y) = (x + cy)\ F(x - y/2) + G(x - y/2); \quad (c \neq 1/2).$
Without loss of generality: $U(x, y) = (x + y/2)\ F(x - y/2) + G(x - y/2).$
Without loss of generality set $c \to 0$ and rescale: $U(x, y) = x\ \tilde{F}(2x - y) + \tilde{G}(2x - y).$ **QED**.

e. $U_{yy} + 2U_{xx} = 0$. **(2 points)**

Solution:

First re-order: $2U_{xx} + U_{yy} = 0$

Discriminant: $h^2 - ab = 0^2 - (2)(1) = -2$. Elliptic.

Alternative 1 — Determinant: $ab - h^2 = (1)(2) - 0^2 = 2$. Elliptic.

Alternative 2 — Determinant: $\det \begin{bmatrix} a & h \\ h & b \end{bmatrix} = \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = 2$. Elliptic.

Quadratic: $a + 2hz + bz^2 = 0 = 2 + 0z + z^2 \quad \Longrightarrow$
$z = (-h \pm \sqrt{h^2 - ab})/b = \pm\sqrt{2}i$.

Apply algorithm: $U(x, y) = F(x + iy\sqrt{2}) + G(x - iy\sqrt{2})$. **QED**.

6. **Euler PDE** (10 points)

Starting with the constant-coefficient Euler PDE

$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} = 0,$$

show that there is a change of independent variables $(x, y) \to (X, Y)$, somewhat different from the change of variables considered in class, such that in terms of the new independent variables

$$U_{XX} + \epsilon\, U_{YY} = 0,$$

where $\epsilon \in \{-1, 0, +1\}$.

Solution:

You can do this in any of *at least* three ways:

- Adapting and modifying the constant-coefficient argument I have already given.
  That is, by first transforming $(x, y) \to (s, t)$, *and then following that by a second transformation* $(s, t) \to (X, Y)$.
- Looking ahead a little in the notes to the variable-coefficient Euler equation, and adapting that argument.
- Or you could just come up with your own proof.

Let's try the first option:

- Using the argument already given in the lectures, transforming $(x, y) \to (s, t)$ yields:
  - If the Euler equation is hyperbolic then $U_{st} = 0$, with s and t real.
    In this case we simply choose the second transformation to be $X = s+t$ and $Y = s-t$ and note that
    $$U_{XX} = U_{ss} + 2U_{st} + U_{tt}$$
    $$U_{YY} = U_{ss} - 2U_{st} + U_{tt}$$
    and so
    $$U_{XX} - U_{YY} = 4U_{st} = 0$$
    and we are done.

– If the the Euler equation is parabolic then $U_{ss} = 0$, with s and t real.
In this case we simply choose the second transformation to be $X = s$ and so $U_{XX} = 0$ and we are done.
– If the Euler equation is elliptic then $U_{st} = 0$ with s and t complex; and note that in this situation s and t are always complex conjugates.
In this case we simply choose the second transformation to be $s = X + iY$ and $t = X - iY$, where X and Y are now real, and note that

$$U_{st} = (\partial_X + i\partial_Y)(\partial_X - i\partial_Y)U = U_{XX} + U_{YY} = 0$$

and we are done.
– That is — collecting the three separate cases — we have

$$U_{XX} + \epsilon\, U_{YY} = 0$$

with $\epsilon = -1$ for hyperbolic, $\epsilon = 0$ for parabolic, and $\epsilon = +1$ for elliptic. **QED**

Let's try the second option:

- Here is another way of doing things: We can rewrite the Euler PDE as

$$(\partial_x, \partial_y) \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} U = 0$$

Concentrate on the matrix

$$E = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

This is a real symmetric matrix so you know you can always diagonalize it using orthogonal transformations (rotations). In the new $(\tilde{x}, \tilde{y})$ coordinate system one has

$$(\partial_{\tilde{x}}, \partial_{\tilde{y}}) \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{pmatrix} \partial_{\tilde{x}} \\ \partial_{\tilde{y}} \end{pmatrix} U = 0$$

where λ_1 and λ_2 are the two eigenvalues of the matrix E.
If both eigenvalues have the same sign the Euler PDE is elliptic; if one of the eigenvalues is zero the Euler PDE is parabolic; if the two eigenvalues have different sign the Euler PDE is hyperbolic. (If both eigenvalues are zero then the original matrix $E = 0$ and the whole PDE s trivial.)
– If the system is elliptic choose $X = \tilde{x}/\sqrt{|\lambda_1|}$ and $Y = \tilde{y}/\sqrt{|\lambda_2|}$, then

$$(\partial_X, \partial_Y) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix} U = 0$$

and we are done.
– If the system is parabolic, re-order the coordinates so that $\lambda_1 \neq 0$ and choose $X = \tilde{x}/\sqrt{\lambda_1}$ and $Y = \tilde{y}$. Then

$$(\partial_X, \partial_Y) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix} U = 0$$

and we are done.

– If the system is hyperbolic, re-order the coordinates so that $\lambda_1 > 0$ and choose $X = \tilde{x}/\sqrt{\lambda_1}$ and $Y = \tilde{y}/\sqrt{|\lambda_2|}$. Then

$$(\partial_X, \partial_Y) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{pmatrix} \partial_X \\ \partial_Y \end{pmatrix} U = 0$$

and we are done.

That is — we have

$$U_{XX} + \epsilon U_{YY} = 0$$

with $\epsilon = -1$ for hyperbolic, $\epsilon = -0$ for parabolic, and $\epsilon = +1$ for elliptic. **QED**

There are many other ways of getting to the same result — these two techniques are just the two most obvious routes.