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MATH 301 | Differential equations | 2024

Tutorial week 4: Frobenius–Mayer

1. **Frobenius–Mayer:**

A vector field V is called conservative iff curl $V = 0$. (Also written as $\nabla \times V = 0$.)

It is a well known fact (Calculus 2) that if V is a conservative vector field on an open subset W of $\mathbb{R}^3$, then there is a function $U(x, y, z)$ such that $V = -\text{grad}\, U$ on W. (Also written as $V = -\nabla U$.)

a. Show that the system of PDEs $\text{grad}\, U = -V$ is a Frobenius system, (a particularly simple Frobenius system), and find the appropriate consistency condition that is needed to apply the Frobenius Complete Integrability theorem.

Solution: Write $\text{grad}\, U = -V$ in the form

$$\frac{\partial U(x,y,z)}{\partial x^i} = -V_i(x,y,z)$$

This is a Frobenius system with one dependent variable, three independent variables, and no U dependence on the RHS — so the system is actually linear as opposed to the more typical quasi-linear case.

Because the RHS has no U dependence, the Frobenius integrability condition reduces to

$$\frac{\partial V_i}{\partial x^j} = \frac{\partial V_j}{\partial x_i} \quad \Longleftrightarrow \quad \partial_i V_j - \partial_j V_i \quad \Longleftrightarrow \quad \epsilon^{ijk}\left[\partial_i V_j - \partial_j V_i\right] = 0$$

$$\Longleftrightarrow \quad \nabla \times \mathbf{V} = 0$$

b. Find the function U if:

i. $\vec{V} = xyz\,\vec{i} + (x^2 z/2 - z\sin(yz))\,\vec{j} + (x^2 y/2 - y\sin(yz))\,\vec{k}$.

Solution: First check the consistency condition

$$\nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ xyz & x^2 z/2 - z\sin(yz) & x^2 y/2 - y\sin(yz) \end{vmatrix}$$

$$= \vec{i}([x^2/2 - \sin(yz) - yz\cos(yz)] - [x^2/2 - sin(yz) - yz\cos(yz)])$$
$$+\vec{j}(-[xy] + [xy]) + \vec{k}([xz] - [xz]) = \vec{0}.$$

So yes, the consistency condition is satisfied.
There are now *many different ways* of calculating $U(x, y, z)$.
One of them is the following: Rewrite the system of PDEs to be solved as

$$\partial_x U = xyz$$

$$\partial_y U = x^2 z/2 - z\sin(yz)$$

$$\partial_z U = x^2 y/2 - y\sin(yz)$$

Integrate each of these PDE's independently, introducing suitable arbitrary functions:

$$U = x^2 yz/2 + f(y, z)$$

$$U = x^2 yz/2 + \cos(yz) + g(x, z)$$

$$U = x^2 yz/2 + \cos(yz) + h(x, y)$$

But all three of these equations must hold *simultaneously*.
Subtracting the last 2 equations we see

$$g(x, z) = h(x, y)$$

which can only be true iff

$$g(x, z) = h(x, y) = j(x)$$

But then subtracting the first 2 equations we have

$$f(y, z) = \cos(yz) + j(x)$$

which can only be true iff

$$f(y, z) = \cos(yz) + k; \qquad j(x) = k$$

whence all 3 equations now agree that

$$U = x^2 yz/2 + \cos(yz) + k$$

You can now check this by differentiating.
[You could also get the same result by performing a line integral along any *arbitrary* path starting at (say) the origin, out to the point (x, y, z).]

ii. $\vec{V} = (A/r^3)\,\vec{r}$.
Here A is some arbitrary but fixed constant, and $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ is the usual radius vector, $\vec{r}$, and $r = |r|$.
Solution: We wish to solve

$$\nabla U = -\vec{V} = -(A/r^3)\vec{r} = -(A/r^2)\hat{r}$$

By inspection

$$U = A/r + C.$$

This is just the usual $1/r$ potential coming from an inverse-square law...