Week 4 Tutorial: Separation of variables

Reminder: In-term test on Tuesday 26 March at 12:00 noon...

1. Attempt to (partially) solve the following PDEs using separation of variables.

   If possible, determine the resulting ODEs.

   (Do not attempt to *solve* the ODEs, just find the variable-separated ODEs, if possible.)

   - (a) $xu_{xx} + u_t = 0$;

     Solution:

     Set $u(x,t) = X(x)T(t)$ so that $u_{xx} = X''T$ and $u_t = XT'$ giving

     $$xX''T + XT' = 0, \qquad \Longrightarrow \qquad \frac{xX''}{X} = -\frac{T'}{T} = k.$$

     Here k is the separation constant, since each side is a function of a different variable.

     So we obtain the separated ODEs:

     $$xX'' - kX = 0, \qquad T' + kT = 0.$$

     Extra:

     The ODE for $T(t)$ has an easy solution $T(t) = A\exp(-kt)$.

     The ODE for $X(x)$ leads to Bessel functions...

   - (b) $tu_{xx} + xu_t = 0$;

     Solution:

     Set $u(x,t) = X(x)T(t)$ so that $u_{xx} = X''T$ and $u_t = XT'$ giving

     $$tX''T + xXT' = 0, \qquad \Longrightarrow \qquad \frac{X''}{xX} = -\frac{T'}{tT} = k.$$

     Here k is the separation constant, since each side is a function of a different variable.

     So we obtain the separated ODEs:

     $$X'' - kxX = 0, \qquad T' + ktT = 0.$$

     Extra:

     These ODEs lead to Airy functions in both space and time directions...

- (c) $u_{xx} + (x+t)u_{tt} = 0$.

  Solution:

  This one gives
  $$X''T + xXT'' + tXT'' = 0.$$
  Since it is *not* possible to reduce this to two terms, each of which factor into a function of x times a function of t, **separation is impossible**.

  The best you could do would be to divide by $u = XT$ to get
  $$\frac{X''}{X} + x\frac{T''}{T} + t\frac{T''}{T} = 0.$$
  But note the middle term still mixes x and t.

  Separation of variables simply does not work for this specific PDE...

- (d) $u_{xx} + u_{tt} + xu = 0$;

  Solution:

  Substituting $U(x,t) = X(x)T(t)$ this one gives
  $$X''T + XT'' + xXT = 0.$$
  Divide by $u = XT$ to get
  $$\frac{X''}{X} + \frac{T''}{T} + x = 0.$$
  Thence
  $$\frac{X''}{X} + x = k = -\frac{T''}{T}.$$
  and
  $$X'' = (k-x)X; \qquad T'' = -kT.$$
  Extra:

  The ODE for $T(t)$ leads to $T(t) = A\exp(i\sqrt{k}t) + B\exp(-i\sqrt{k}t)$.

  The ODE for $X(x)$ leads to shifted Airy functions... Use $\tilde{x} = x - k$.

2. The heat equation for $u(x,y,t)$ in two spatial dimensions has the form
$$\sigma^2(u_{xx} + u_{yy}) = u_t$$
If $u(x,y,t) = X(x)Y(y)T(t)$ find ODEs for $X(x)$, $Y(y)$, and $T(t)$.

(Do not attempt to *solve* the ODEs, just find the variable-separated ODEs.)

Solution:

We have $u_{xx} = X''YT$, $u_{yy} = XY''T$, $u_t = XYT'$.

So
$$\sigma^2(X''YT + XY''T) = XYT' \qquad \Longrightarrow \qquad \sigma^2(X''Y + XY'')T = XYT'.$$

By dividing by XYT we can first separate T to get

$$\frac{\sigma^2(X''Y + XY'')}{XY} = \frac{T'}{T} = k.$$

Here the left-hand side is a function of x, y and the right-hand side of t hence they both must be constant.

Now we have

$$\frac{\sigma^2(X''Y + XY'')}{XY} = k \qquad \Longrightarrow \qquad \frac{\sigma^2 X''}{X} + \frac{\sigma^2 Y''}{Y} = k.$$

But this means

$$\frac{\sigma^2 X''}{X} = q_1; \qquad \frac{\sigma^2 Y''}{Y} = q_2; \qquad q_1 + q_2 = k.$$

Thence we have the 3 equations

$$\begin{aligned} \sigma^2 X'' - q_1 X &= 0; \\ \sigma^2 Y'' - q_2 Y &= 0; \\ T' - (q_1 + q_2)T &= 0. \end{aligned}$$

Extra:

The ODE for $T(t)$ leads to $T(t) = A\exp([q_1 + q_2]t)$.

The ODE for $X(x)$ leads to $X(x) = B\exp(\sqrt{q_1}x) + C\exp(-\sqrt{q_1}x)$.

The ODE for $Y(y)$ leads to $Y(y) = D\exp(\sqrt{q_2}x) + E\exp(-\sqrt{q_2}x)$.

Consequently [before applying any boundary conditions]

$$u(x,y,t) = \int\int [B(q_1,q_2)\exp(\sqrt{q_1}x) + C(q_1,q_2)\exp(-\sqrt{q_1}x)]$$
$$[D(q_1,q_2)\exp(\sqrt{q_2}x) + E(q_1,q_2)\exp(-\sqrt{q_2}x)]\exp([q_1+q_2]t)\ dq_1 dq_2.$$

3. Using separation of variables, find a solution to Laplace's equation $u_{xx} + u_{yy} = 0$ on the rectangle $0 < x < a$, $0 < y < b$ with Dirichlet boundary conditions:

$$\begin{aligned} &u(0,y) = 0, &&u(a,y) = f(y), &&0 < y < b; \\ &u(x,0) = g(x), &&u(x,b) = 0, &&0 \le x \le a. \end{aligned}$$

Note *two* edges are non-zero.
Hint: Consider adding the solutions to 2 simpler problems.

Solution:

Consider these two simpler problems

$$\begin{aligned} &u_1(0,y)=0, && u_1(a,y)=0, && 0<y<b;\\ &u_1(x,0)=g(x), && u_1(x,b)=0, && 0\le x\le a; \end{aligned}$$

and

$$\begin{aligned} &u_2(0,y)=0, && u_2(a,y)=f(y), && 0<y<b;\\ &u_2(x,0)=0, && u_2(x,b)=0, && 0\le x\le a; \end{aligned}$$

and then consider

$$u(x,y)=u_1(x,y)+u_2(x,y).$$

Each of the 2 sub problems is “simple”.

Extra:

Using separation of variables, find a solution to Laplace's equation $u_{xx}+u_{yy}=0$ on the rectangle $0<x<a$, $0<y<b$ with Dirichlet boundary conditions:

$$\begin{aligned} &u(0,y)=j(y), && u(a,y)=f(y), && 0<y<b;\\ &u(x,0)=g(x), && u(x,b)=k(x), && 0\le x\le a. \end{aligned}$$

Note all *four* edges are non-zero.

4. Using separation of variables, find the solution to Laplace's equation in the semi-infinite strip $0<x<a$, $y>0$ with boundary conditions

$$\begin{aligned} &u(0,y)=0, && y>0;\\ &u(a,y)=0, && y>0;\\ &u(x,0)=f(x), && 0\le x\le a;\\ &\lim_{y\to\infty}u(x,y)=0 && 0<x<a. \end{aligned}$$

Solution:

First try to separate variables:

$$U(x,y)=X(x)Y(y).$$

Then Laplace's equation becomes

$$X''(x)Y(y)+X(x)Y''(y)=0$$

Thence, dividing by $X(x)Y(y)$ we have

$$\frac{X''(x)}{X(x)}+\frac{Y''(y)}{Y(y)}=0$$

Thence we have a separation constant

$$\frac{X''(x)}{X(x)}=-k;\qquad \frac{Y''(y)}{Y(y)}=+k.$$

We have 3 possibilities: $k<0$, $k=0$, $k>0$.

- $k < 0$:
  Then $k = -b^2$ and $X''(x) = b^2 X(x)$ so $X(x) = A\cosh(bx) + B\sinh(bx)$;
  but then the boundary conditions in x imply $X(0) = 0 = X(b)$,
  which in turn implies $X(x) \equiv 0$,
  which is uninteresting.
- $k = 0$:
  Then $X''(x) = 0$ so $X(x) = A + Bx$;
  but then the boundary conditions in x imply $X(0) = 0 = X(b)$,
  which in turn implies $X(x) \equiv 0$,
  which is uninteresting.
- $k > 0$:
  Then $k = +b^2$ and $X''(x) = -b^2 X(x)$ so $X(x) = A\cos(bx) + B\sin(bx)$;
  but then the boundary conditions in x imply $X(0) = 0 = X(b)$,
  which in turn implies $A = 0$ and $\sin(ba) = 0$,
  so $b = n\pi/a$ and $X(x) = B\sin(n\pi x/a)$.
  But now $Y''(y) = +b^2 Y(y)$ with $b > 0$, so $Y(y) = C\exp(by) + D\exp(-by)$;
  but then the asymptotic boundary condition in y implies $Y(\infty) = 0$,
  which in turn implies $C = 0$.

  At this stage we have

$$U(x, y) = X(x)Y(y) = B\sin(n\pi x/a)\ D\exp(-n\pi y/a).$$

Invoking linear superposition

$$U(x, y) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a)\exp(-n\pi y/a).$$

This satisfies Laplace's equation and the *three* homogeneous boundary conditions.

The only remaining condition is $U(x, 0) = f(x)$ which implies

$$f(x) = \sum_{n=1}^{\infty} E_n \sin(n\pi x/a).$$

This in principle determines the E_n and we are done.
