MATH301: Differential Equations
Partial Differential Equations

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Nonuniform transport

Consider the problem

$$u_t + c(t,x)u_x = 0$$

with

$$u(0,x) = f(x)$$

- Here the speed is not constant $c(t,x)$
- We will work same as before but there is not guarantee that we can solve the problem
- The solvability depends on $c(t,x)$

Nonuniform transport

Take $u(t, x(t))$ the solution of our problem along the lines that satisfy

$$\frac{dx(t)}{dt} = c(t, x(t))$$

This means

$$\begin{aligned}\frac{d}{dt}u(t, x(t)) &= u_t(t, x(t)) + x'(t)u_x(t, x(t)) \\ &= u_t(t, x(t)) + c(t, x(t))u_x(t, x(t)) \\ &= 0\end{aligned}$$

- The method of characteristics works again!
- Only if we can solve the equation $x'(t) = c(t, x(t))$ which can be nonlinear!

Nonuniform transport

Solve

$$x'(t) = c(t, x(t))$$

with $x(0) = k \in \mathbb{R}$ then the solution is constant along $x(t)$ thus

$$\begin{aligned} u(t, x(t)) &= u(0, x(0)) \\ &= u(0, k) \\ &= f(k) \end{aligned}$$

We can find solution only if we can find k as a function of $x(t)$.

Nonuniform transport

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The transport equation

Example

Solve the Initial Value Problem

$$u_t + \frac{1}{x^2+1} u_x = 0 \ ,$$

when

$$u(0,x) = \frac{1}{1+(x+3)^2} \ .$$

Solution

First solve

$$x'(t) = \frac{1}{x^2(t) + 1}$$

or easier

$$\frac{dx}{dt} = \frac{1}{x^2 + 1}$$

This is separation of variables

$$\int (x^2 + 1) dx = \int 1 dt$$

Thus

$$\frac{1}{3}x^3 + x = t + \frac{1}{3}k^3 + k$$

which is hard to solve for k

The transport equation

Example

Solve the Initial Value Problem

$$u_t + tu_x = 0 \ ,$$

when

$$u(0,x) = e^{-x^2} \ .$$

Solution

First solve

$$x'(t) = t$$

This is a separable equation with

$$x = \frac{1}{2}t^2 + k$$

Thus

$$k = x - \frac{1}{2}t^2$$

and the solution is

$$u(x,t) = e^{-(x-\frac{1}{2}t^2)^2}$$

Nonlinear waves

Let's study the nonlinear equation

$$u_t + uu_x = 0$$

with initial data $u(0, x) = f(x)$.
Again we consider characteristic lines to be the solutions to the following IVP:

$$\begin{cases} x'(t) = u(t, x(t)), \text{ for } t > 0 \\ x(0) = k \text{ given number} \end{cases}$$

Nonlinear waves

Let's denote $h(t) = u(t, x(t))$ the solution along a characteristic line $x = x(t)$. Then,

$$
\begin{aligned}
h'(t) &= \frac{d}{dt}u(t, x(t)) \\
&= u_t(t, x(t)) + x'(t)u_x(t, x(t)) \\
&= u_t(t, x(t)) + u(t, x(t))u_x(t, x(t)) \\
&= u_t + uu_x \qquad \text{for } x = x(t) \\
&= 0
\end{aligned}
$$

Thus, the solution $u(t, x)$ is constant along the characteristics! Miracle!

Nonlinear waves

Since $\frac{d}{dt}u(t,x(t))=0$ it implies

$$u(t,x(t))=\text{constant}=u(0,x(0))=u(0,k)=f(k)$$

Going back to the characteristic equation we have

$$x'(t)=u(t,x(t))=f(k)$$

Since $f(k)$ is constant we have after integration between 0 and t that

$$x(t)=f(k)t+k,\quad \text{for } t>0.$$

Thus the characteristic lines are straight lines

Nonlinear waves

If we were able to solve $x = f(k)t + k$ with respect to k then we can substitute into the solution

$$u(t, x) = f(k)$$

and get the explicit solution. Because $f(k)$ might be nonlinear, the situation can be complicated and we can ended up solving nonlinear equations with computers and approximately.

Characteristic lines

Note that the slope of the characteristic lines is $f(k)$ that thus each characteristic line has different slope that depends on the initial condition $f(k)$. This can cause characteristic lines to intersect or to become distant, depending on the function $f(k)$.

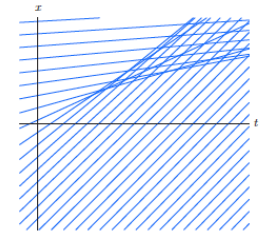
Nonlinear waves

Example

Consider the equation $u_t + uu_x = 0$ with initial condition

$$u(0,x) = f(x) = \frac{1}{2}\pi - \tan^{-1} x$$

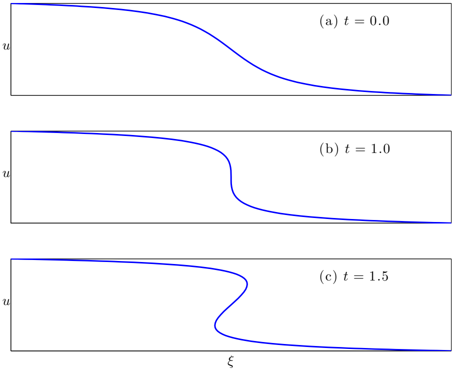
The characteristic lines satisfy the equation $x(t) = f(k)t + k$ depicted bellow in the $t-x$-plane



Nonlinear waves

Example

The solution breaks at $t = 1$ and then becomes multi-valued!



Nonlinear waves

Let's solve a problem until wave breaking

Example

Consider $u_t + uu_x = 0$ for $x \in \mathbb{R}$ and $t > 0$ with initial condition

$$u(t,0) = f(x) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

We have the characteristic curves

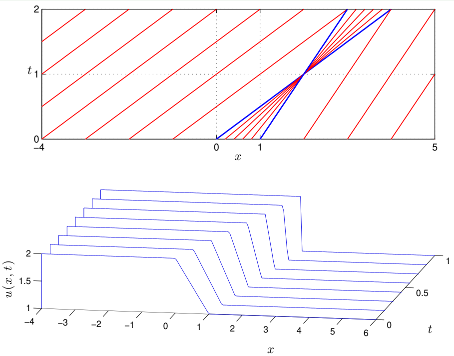
$$x(t) = f(k)t + k = \begin{cases} 2t + k, & k < 0 \\ (2-k)t + k, & 0 \leq k \leq 1 \\ t + k, & k > 1 \end{cases}$$

Nonlinear waves

Example

The solution then is

$$u(t,x) = \begin{cases} 2, & x-2t<0 \\ \frac{2-x}{1-t}, & 0 \le \frac{x-2t}{1-t} \le 1 \\ 1, & x-t>1 \end{cases}$$



Wave breaking

Breaking time

We define the breaking time to be the minimum time $t^* > 0$ for which the solution has infinite gradient, thus

$$u_x(t,x) \to \infty \quad \text{as} \quad t \to t^*$$

Wave breaking

Along the characteristic lines $x = f(k)t + k$ we have $u(t, x) = f(k)$.

$$\begin{aligned} u_x(t,x) &= \frac{d}{dx} f(k) \\ &= k_x f'(k) \end{aligned}$$

Moreover, differentiating with respect to x the equation $x = f(k)t + k$ we get

$$1 = k_x f'(k)t + k_x$$

which gives

$$k_x = \frac{1}{1 + f'(k)t}$$

Substitution into u_x gives that along the characteristic lines

$$u_x(t,x) = \frac{f'(k)}{1 + f'(k)t} \ .$$

Wave breaking

Substitution into u_x gives that along the characteristic lines

$$u_x(t,x) = \frac{f'(k)}{1+f'(k)t} \ .$$

This becomes infinite only when the denominator becomes 0. This means that $t = -\frac{1}{f'(k)}$ with $k \in \mathbb{R}$. But the breaking time is defined to be the first instance so we have:

$$t^* = \min_k \left\{ -\frac{1}{f'(k)} \right\} \ .$$

If $g(k) = -\frac{1}{f'(k)}$ how to find minimum values?

Wave breaking

In our previous example the wavebreaking occurs with $f(k) = 2 - k$ thus we need to find the extrema of the function $g(k) = 1$ and thus the breaking time is $t^* = 1$.