School of Mathematics and Statistics

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MATH 301 **Assignment 3**

Due Date: Friday 10th September before 5pm.

1. Find the general solution $u(t, x)$ to the following PDEs:
   (a) $\frac{\partial^2 u}{\partial t^2} = 0$
   (b) $u_x + tu = 0$

2. Solve the PDE $u_t + 2tx^2 u_x = 0$.

3. Give the order and classify the following equations as (i) homogeneous linear, (ii) inhomogeneous linear, or (iii) nonlinear:
   (a) $u_t + uu_x = 3u$
   (b) $u_x(1 - u_x^2)^{-1/2} + u_y(1 + u_y^2)^{-1/2}$
   (c) $u_{xxx} - \cos u = u_t$
   (d) $\nabla \cdot u = 0$
   (e) $u_{xx}u_{yy} - u_{xy}^2 = 0$

4. (a) Find the general solution to the PDE $u_t + \frac{3}{2}u_x = 0$
   (b) Find a specific solution with the initial condition $u(0, x) = \sin x$. Is your solution unique?

5. Consider the initial value problem

$$u_t + uu_x = 0, \quad x \in \mathbb{R} \quad t > 0,$$
$$u(x, 0) = e^{-x^2}, \quad x \in \mathbb{R}.$$

   Sketch the characteristic diagram and find the point (x_b, t_b) in space-time where the wave breaks.

6. By writing $v = u_y$, solve the equation $3u_y + u_{xy} = 0$ subject to the condition $u_x(0, y) = 1$ and $u(x, 0) = x$.

7. Find the linear dispersion relation of the Benjamin-Bona-Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \qquad x \in (-\infty, +\infty) .$$

   Prove that the solution of the BBM equation given any initial condition $u(0, x) = u_0(x)$ preserve the following quantities (functionals):

$$M(t; u) = \int_{-\infty}^{\infty} u(t, x)\, dx \qquad \texttt{(Mass)}$$

$$I(t; u) = \int_{-\infty}^{\infty} u^2(t, x) + u_x^2(t, x)\, dx \qquad \texttt{(Impulse)}$$

$$E(t; u) = \int_{-\infty}^{\infty} u^2(t, x) + \tfrac{1}{3}u_x^3(t, x)\, dx \qquad \texttt{(Energy)}$$

   Find an analytical formula for the solitary waves solution of the BBM equation that travels with constant speed $c_s > 0$.