SCHOOL OF MATHEMATICS AND STATISTICS
*Te Kura Mātai Tatauranga*

MATH 301 DIFFERENTIAL EQUATIONS (CALCULUS 3) T1 2024

# Math 301 — Partial Differential Equations
# Part I: Weeks 1 to 6 (Draft)

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**Warning:**

These notes are provided as a *supplement* to the lectures.
They are *not* a substitute for attending the lectures.
I'm sure there are typos — let me know.

# Contents

# Chapter 1
# Introduction

- **Math 301, *Partial Differential Equations*, PDEs**
- Version: Autumn 2024. (Draft.)
- Part 1 of the course consists of a brief introduction to PDEs.
- Part 2 will cover a few more specialized topics.
- PDEs are one of the most useful tools of applied mathematics and mathematical physics. If you intend to continue studying in either of these fields, get used to working with PDEs — they are ubiquitous.
- PDEs are also central to mathematical finance, where they underlie (for instance) the Black–Scholes theory for the pricing of stock market options and [financial] derivatives.
- This set of notes is rather roughly based on an older collection of notes originally provided some 25 years ago by Dr Chris Grigson; with various modifications due to Professor Mark McGuinness, and myself.
- Updates, LaTeX conversion, corrections, and extensive additions have been made by Matt Visser.
- Modern textbook (for background reference):
  Peter J Olver,
  *Introduction to Partial Differential Equations*,
  Springer.
- These lectures correspond very roughly to Chapters 1 to 4 of Olver.
- Note:
  ODE = Ordinary Differential Equation;
  PDE = Partial Differential Equation.

In addition to the "official" textbook, and this set of notes, there are *many* other books you can look at for additional background material, ideas, and examples. A traditional textbook is:

- Boyce and DiPrima,
  *Elementary Differential Equations and Boundary Value Problems*,
  Seventh edition.
  These lecture notes correspond very roughly to Chapter 10:
  *Partial differential equations and Fourier series.*

Other good solid textbooks include:

- Erwin Kreyszig,
  "Advanced engineering mathematics".
- Stanley Farlow,
  "Partial differential equations for scientists and engineers".
- Ronald Guenther and John Lee,
  "Partial differential equations of mathematical physics and integral equations".
- Carl Bender and Steven Orszag,
  "Advanced mathematical methods for scientists and engineers".
- Ray Wylie and Louis Barrett,
  "Advanced engineering mathematics".
- Dennis Zill and Michael Cullen,
  "Differential equations with boundary value problems".
- Kent Nagle and Edward Saff,
  "Fundamentals of differential equations".
- Yehuda Pinchover and Jacob Rubinstein,
  "An introduction to partial differential equations".
- S. L. Sobelov,
  "Partial differential equations of mathematical physics".
- E. C. Zachmanoglou and Dale Thoe,
  "Introduction to partial differential equations with applications".
- K. F. Riley, M. P. Hobson, and S. J. Bence,
  "Mathematical methods for physics and engineering".
- Walter Strauss,
  "Partial differential equations: An introduction".
- Robert Borrelli and Courtney Coleman,
  "Differential equations: A modelling perspective".

- Polyanin's "handbook" series:
  - Andrei Polyanin and Valentin Zaitsev,
    "Exact solutions for ordinary differential equations".
  - Andrei Polyanin and Valentin Zaitsev,
    "Nonlinear partial differential equations".
  - Andrei Polyanin,
    "Linear partial differential equations for scientists and engineers".
  - Andrei Polyanin, Valentin Zaitsev, and A. Moussiaux,
    "First order partial differential equations".
- In addition, Google can quite easily direct you to *lots* of online notes on PDEs — almost all of very high quality.
- This is also a topic on which Wikipedia is reasonably trustworthy.
- See for instance:
  - http://en.wikipedia.org/wiki/Partial_differential_equation
  - http://en.wikipedia.org/wiki/First_order_partial_differential_equation
  - http://en.wikipedia.org/wiki/Separable_partial_differential_equation
  - http://en.wikipedia.org/wiki/Separation_of_variables
  - http://en.wikipedia.org/wiki/Method_of_characteristics
  - http://en.wikipedia.org/wiki/Fourier_series
  - http://en.wikipedia.org/wiki/Convergence_of_Fourier_series

# Chapter 2

# Fundamentals

## 2.1 Basic definition of a PDE

**Definition 1 PDEs:**
*A partial differential equation (PDE) is an equation involving one or more unknown functions, (the "fields"), of two or more independent variables, ("position" and possibly "time"), and the derivatives of the unknown functions with respect to the independent variables.*

## 2.2 Variables (independent and dependent)

- For simplicity we shall generally consider there to be *two* independent variables, denoted either by $x$ and $y$, by $t$ and $x$, (sometimes [rarely] by $x_1$ and $x_2$), or by $x^1$ and $x^2$.
- In differential geometry and most of theoretical physics it is typically most common to use *superscripts* to denote the different independent variables, $x^1$ and $x^2$.
- A potential problem with this convention is that you then you have to be careful to not get confused with exponents.

$$\text{That is:} \qquad x^2 \neq (x)^2 \; ! \tag{2.1}$$

- Nevertheless the superscript convention is so well established [in both applied mathematics and theoretical physics] that I will consistently adopt it throughout these notes. So get used to seeing things like $x^1$ and $x^2$.
- Notation such as $x_1$ and $x_2$ is to be discouraged.
  (You might sometimes still see such notation — just don't copy it.)
- The generalization of results and methods to more than two independent variables will be "straightforward" and is left to you.
- (Actually "straightforward" is a "code word" that you should learn to recognize — it means that extensions to more than two dimensions are in principle easy but in practice can turn quickly into computational nightmares.)
- This means that almost everything we will be doing is either in $(1+1)$ dimensions [one space dimension, plus one time dimension] or in two space dimensions — $(2+0)$ dimensions if you want to be difficult.
- Some constructions and techniques do depend specifically on the number of dimensions — watch out; I'll try to give you appropriate warnings.
- There are some features of $(3+1)$ dimensions [three space dimensions, plus one time dimension], the universe we live in, that are just not adequately captured by the $(1+1)$ dimensional simplification.
- In cases of high symmetry one can often reduce the effective number of dimensions of the problem:
  - Spherical symmetry in (3+1) dimensions
    $\Longrightarrow$ everything depends on at most distance from the centre,
    and perhaps on time,
    $\Longrightarrow$ effectively (1+1) dimensions.
  - Cylindrical symmetry in (3+1) dimensions
    $\Longrightarrow$ everything depends on at most distance from the axis,
    and perhaps on time,
    $\Longrightarrow$ effectively (1+1) dimensions.
  - Planar symmetry in (3+1) dimensions
    $\Longrightarrow$ everything depends $x$ and $y$ but not $z$,
    and perhaps on time,
    $\Longrightarrow$ effectively (2+1) dimensions.
  - Time independent in (3+1) dimensions
    $\Longrightarrow$ effectively (3+0) = 3 dimensions.

- We shall also [for many of these lectures, excluding the section on Frobenius systems] assume there is only one dependent variable.
  (In physics language, we are dealing with only one "field"; such as pressure, or density, or displacement.)
  We shall use any of the symbols $U$, $u$, $V$, $v$ ... to denote that variable.
- The generalization of results and methods to more than one dependent variable will be "straightforward" and is left to you.
- (Notice that code word "straightforward" again. Be very afraid.)
- Physically, generalizing to more than one dependent variable would be useful in situations such as:
  - Electric and magnetic fields [the Maxwell equations], 6 "fields", in (3+1) dimensions.
  - Einstein's theory of gravity [the general relativity, where there are 10 inter-connected gravitational "potentials"], (3+1) dimensions.
  - Fluid mechanics [where at a minimum you have to keep track of both density and velocity].

  Still, one step at a time, in this course we will mostly stick to *one* dependent variable.
- Warning: you will soon se that the mathematical theory of (general) PDEs is much less well-developed than the mathematical theory of general ODEs.
- When it comes to PDEs, the mathematical situation is still pretty much that we have a lot of information about a large number of special cases — and relatively little information about truly general situations.

## 2.3 Partial derivatives

There are *many* different notations used for partial derivatives.
Variously used, but *completely equivalent*, notations are:

$$
\begin{aligned}
D_{(0,1)}U &= D_1U = D_xU = \frac{\partial U}{\partial x} = \partial_x U = U_x = \frac{\partial U}{\partial x_1} = \frac{\partial U}{\partial x^1}. && (2.2)\\
D_{(1,0)}U &= D_2U = D_yU = \frac{\partial U}{\partial y} = \partial_y U = U_y = \frac{\partial U}{\partial x_2} = \frac{\partial U}{\partial x^2}. && (2.3)\\
D_{(1,1)}U &= D_1D_2U = D_xD_yU = \frac{\partial^2 U}{\partial x\, \partial y} = \partial_x\partial_y U = U_{xy} && (2.4)\\
&= \frac{\partial^2 U}{\partial x_1\, \partial x_2} = \frac{\partial^2 U}{\partial x^1\, \partial x^2}. && (2.5)\\
D_{(2,1)}U &= D_1^2D_2U = D_x^2D_yU = \frac{\partial^3 U}{(\partial x)^2\, \partial y} = \partial_x^2\partial_y U = U_{xxy} && (2.6)\\
&= \frac{\partial^3 U}{(\partial x_1)^2\, \partial x_2} = \frac{\partial^3 U}{(\partial x^1)^2\, \partial x^2}. && (2.7)
\end{aligned}
$$

And so on. ...

*Learn to recognize all of these variant notations.*

I will try to *standardize* notation in this course to be as follows:

$$
\begin{aligned}
U_x &= \partial_x U = \frac{\partial U}{\partial x}. && (2.8)\\
U_y &= \partial_y U = \frac{\partial U}{\partial y}. && (2.9)\\
U_{xy} &= \partial_x\partial_y U = \frac{\partial^2 U}{\partial x\, \partial y}. && (2.10)\\
U_{xxy} &= \partial_x^2\partial_y U = \frac{\partial^3 U}{(\partial x)^2\, \partial y}. && (2.11)
\end{aligned}
$$

I will also *sometimes* use:

$$U_{,i} = \partial_i U = \frac{\partial U}{\partial x^i}, \tag{2.12}$$

especially when I have more than two independent variables to deal with.

These "standard" notations are the *most common* of the notations you are likely to run into when reading books or scientific articles.

**Notes:**

1. So long as the function $U(x, y)$ is $C^s$ (meaning that all partial derivatives up to order $s$ exist and are continuous), then the sequence in which you take the partial derivatives in an $r$-th order derivative, for any $r \leq s$, does not matter.
2. In the usual spirit of applied mathematics and theoretical physics, we shall take all our functions to be *smooth enough*, in the sense that all partial derivatives that we may happen to need will be assumed to exist and to be continuous.
3. That is, *for all practical purposes*:

$$\boxed{\frac{\partial^2 U}{\partial x\, \partial y} = \frac{\partial^2 U}{\partial y\, \partial x}.} \tag{2.13}$$

## 2.4 Order

**Definition 2 Order:**
*The* **order** *of a PDE is the highest order of differentiation appearing in the PDE.*
*Do not confuse this with the* **degree** *of terms appearing in the equation.*

- If we wish to refer to a general derivative of $U$ of the $m$-th order, without regard to the precise variables that are being used in the differentiation, we shall write $U^{(m)}$.

- That is, $U^{(m)}$ stands generically for an $m$-th order derivative, and we can write
$$F\left(x, y, U, U^{(1)}, U^{(2)}, ...., U^{(n)}\right) = 0 \tag{2.14}$$
as the general form of an $n$-th order PDE, with one dependent variable $U$, and two independent variables $x$ and $y$.
- Note that $U^{(2)}$ for instance could mean any (or all) of $U_{xx}$, $U_{xy}$, $U_{yy}$.

**Definition 3 $n$-th order PDE:**
*An n-th order PDE is a relation of the form*
$$F\left(x, y, U, U^{(1)}, ..., U^{(n)}\right) = 0. \tag{2.15}$$

Order is a statement about how many times you will need to differentiate the dependent variable to even write down the PDE.

## 2.5 Linearity

**Definition 4 Linear PDE:**
*An n-th order PDE,*
$$F\left(x, y, U, U^{(1)}, ..., U^{(n)}\right) = 0, \tag{2.16}$$
*is n-th order linear if it is of the form:*
$$a_n(x, y)\ U^{(n)} + a_{n-1}(x, y)\ U^{(n-1)} + ... + a_0(x, y)\ U + b(x, y) = 0, \tag{2.17}$$
*with $a_n(x, y)$ not identically zero.*

*If it is linear, then it is homogeneous if $b(x, y) = 0$.*

Linearity is a statement about the manner in which the dependent field $U(x, y)$ appears in the PDE.

**Examples:**

- The wave equation:
$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = 0 \tag{2.18}$$
is a second-order linear homogeneous equation.
- The Klein–Gordon equation:
$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = m^2 U \tag{2.19}$$
is a second-order linear homogeneous equation.
- The Sine–Gordon equation:
$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = \sin U \tag{2.20}$$
is a second-order non-linear equation.
- The Korteweg–deVries (KdV) equation:
$$\frac{\partial^3 U}{\partial x^3} + 6U \frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0. \tag{2.21}$$
This equation arises as one particular model for describing shallow water waves.

  It is a third-order non-linear PDE.
- Both KdV and SG have become very prominent as model equations for analyzing problems involving solitary waves (solitons).
- Laplace's equation:
$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \tag{2.22}$$
is a second-order linear equation which is important in the description of many electrostatic and gravitational phenomena. (Newtonian gravity that is, not general relativity.)
- The diffusion equation (heat equation):
$$\frac{\partial^2 U}{\partial x^2} - \frac{\partial U}{\partial t} = 0, \tag{2.23}$$
is an important second-order linear equation which describes many transfer problems, such as diffusion (gaseous or chemical) or heat transfer.

- The Boltzmann equation of statistical mechanics is an equation of this type.
- The diffusion equation is also important in various "random walk" models ("drunkard's walk"), and underlies important financial mathematics in the theory of financial derivative pricing — the Black–Scholes differential equation is of this type.
- Similarly "genetic drift" in population dynamics is governed by diffusion-type equations.

- The (free) Schroedinger equation:

$$\frac{\partial^2 U}{\partial x^2} - i\frac{\partial U}{\partial t} = 0, \tag{2.24}$$

  is a complexified version of the diffusion equation. It is again second-order, and linear, and homogeneous.

  This equation underlies all of quantum physics, and a good hunk of modern technology (in particular, all solid state electronics).
- The Maxwell equations of classical electromagnetism are coupled first-order linear inhomogeneous PDEs in 6 dependent variables (the electric and magnetic fields) and 4 independent variables (space + time).
- Linear PDEs are extremely useful, and quite a lot is known about them.
- *Much less is known about general nonlinear PDEs.*

## 2.6 Quasi-linearity

**Definition 5 Quasi-linear PDE:**
*An n-th order PDE,*

$$F\left(x, y, U, U^{(1)}, ..., U^{(n)}\right) = 0, \tag{2.25}$$

*is quasi-linear if it is linear in the n-th order derivatives.*

*(It is allowed to be nonlinear in lower-order derivatives, and even the coefficients of the n-th order derivatives are allowed to depend on the lower-order derivatives in a nonlinear manner).*

*That is, letting $U_A^{(n)}$ denote the various possible n-th order derivatives, a quasi-linear PDE is described by an equation of the form*

$$\sum_A C^A(x, y, U, U^{(1)}, ..., U^{(n-1)})\, U_A^{(n)} + \tilde{F}\left(x, y, U, U^{(1)}, ..., U^{(n-1)}\right) = 0, \tag{2.26}$$

*with the $C^A(x, y, U, U^{(1)}, ..., U^{(n-1)})$ not all identically zero.*

**Examples:**

- First-order quasi-linear PDEs are of the form

$$\alpha(x, y, U)\, \partial_x U + \beta(x, y, U)\, \partial_y U + \gamma(x, y, U) = 0. \tag{2.27}$$

  Quite a lot is still known about solving PDEs of this type.
  (There is technique called the method of characteristics, which we will at best only mention in this part of the course.)
  Specific examples:

  - $xu_x + (x + y)u_y = u + 1$.
  - $xu_x + u^4 u_y = u^3$.

- Second-order quasi-linear PDEs are of the form

$$a(x, y, U, U_x, U_y)\, U_{xx} + b(x, y, U, U_x, U_y)\, U_{xy} + c(x, y, U, U_x, U_y)\, U_{yy} + d(x, y, U, U_x, U_y) = 0. \tag{2.28}$$

  We shall see these PDEs again later on in the course, and under a different name.

- The Sine–Gordon equation:

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = \sin U \tag{2.29}$$

  is a specific second-order quasi-linear equation.
- The quasi-linear Klein–Gordon equation

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} + m^2\, U = \lambda\, U^3 \tag{2.30}$$

  is another commonly occurring second-order quasi-linear equation.

  (**Challenge:** Find the *exact* plane wave solutions for this PDE, and no, I am not talking about sine and cosine anymore...)

  (**Challenge:** Relate this PDE to the Higgs particle occurring in the standard model of particle physics.)
- Mathematicians now quite often talk about "$f$–Gordon equations" where

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial^2 U}{\partial x^2} = f(x, t, U, U_x, U_y). \tag{2.31}$$

  Here $f(x, t, U, U_x, U_y)$ is an *arbitrary* nonlinear function of its arguments.
- The Korteweg–deVries (KdV) equation

$$\frac{\partial^3 U}{\partial x^3} + 6U\,\frac{\partial U}{\partial x} + \frac{\partial U}{\partial t} = 0, \tag{2.32}$$

  is quasi-linear because the $U_{xxx}$ term occurs linearly.
- Keeping the highest-order derivatives linear is sometimes enough to let us prove useful theorems.
- Quite a lot (comparatively speaking) is known about quasi-linear PDEs.
- The Einstein equations of classical general relativity are second-order quasi-linear PDEs in 10 dependent variables (the "metric" describing the spacetime geometry) and 4 independent variables (space + time). (General relativity is however nonlinear in the first-order derivatives and non-linear in the metric components themselves.)
- PhD theses are still being written on (advanced) first-order *systems* of PDEs.
- PhD theses are still being written on (advanced) second-order PDEs.

## 2.7 Boundary conditions/Initial conditions

Many PDEs arise in problems for which, in addition to defining the PDE to solve, there are naturally occurring conditions, called "boundary conditions" (BC), [or sometimes "initial conditions" (IC)], that the solution must also satisfy.

In many common specific cases, the PDEs and their associated BC and/or IC can be classified into standard types (with names such as, "elliptic", "hyperbolic", "parabolic") for which the whole problem, PDE and associated BC and/or IC, can be shown to have a unique solution.

The distinction between boundary conditions and initial conditions makes sense only if you have a problem involving both space and time.

- **Initial conditions** provide constraints on the dependent variables at some initial instant in time, throughout some region of space.
- **Boundary conditions** provide constraints on the dependent variables at some place in space, throughout some interval of time.
- **Radiation conditions** provide constraints on the dependent variables in terms of incoming [or outgoing] wave motion.

If you are into special or general relativity — Initial conditions are specified on spacelike surfaces, boundary conditions are specified on timelike surfaces, and radiation conditions are specified on lightlike surfaces [null surfaces].

And to add confusion, sometimes the phrase "boundary conditions" is used *indiscriminately to refer to all three types.*

Suppose now we denote the boundary by the curve $(x(s), y(s))$, or more generally the surface $\vec{x}(\sigma)$, and denote the normal derivative to the boundary by $\partial_n$. Standard terminology is:

- **Normal derivative:**
$$\partial_n = \hat{\mathbf{n}} \cdot \nabla \tag{2.33}$$
- **Dirichlet BC:** The value of the dependent variable is specified on the boundary:
$$U(\vec{x}(\sigma)) = f(\sigma). \tag{2.34}$$
- **Neumann BC:** The value of the normal derivative of the dependent variable is specified on the boundary:
$$\partial_n U(\vec{x}(\sigma)) = f(\sigma). \tag{2.35}$$

- **Robin BC:** Some linear combination of the dependent variable and its normal derivative is specified on the boundary:

$$a(\sigma)\, U(\vec{x}(\sigma)) + b(\sigma)\, \partial_n U(\vec{x}(\sigma)) = f(\sigma). \tag{2.36}$$

There is a vast literature on solving equations of these types.
Look, for example, in:

- Courant, R. and D. Hilbert,
  *Methods of Mathematical Physics Vols 1 and 2.*

*I'll have a lot more to say about these issues soon.*

## 2.8 Exercises (On order, linearity, etc...)

**Reminder:**

- The order of a PDE is the order of the highest derivative appearing in the equation.
- PDE is linear if it is of the first degree in the dependent variables and their derivatives.
- A linear PDE is homogeneous if every term in its expression is linear in the dependent variables and their derivatives.
- A PDE is quasi-linear if the highest-order "derivative part" is linear, though the coefficients and the sub-leading terms are allowed to be nonlinear.
- If a PDE is nonlinear the question of whether or not it is homogeneous is best regarded as meaningless.

Classify the following PDEs:

- By stating their order.
- By stating whether they are linear or not linear.
- If linear, classify then as to whether they are homogeneous or not.
- If nonlinear, classify then as to whether they are quasi-linear or not.

a. $V^2\, V_{xy} + V_x\, V_y + (x^2 - y^2)V = 3xy$.

b. $U_{xxz} - 2(x+z)U_{xyz} - U_{xx} + \sin(xyz)U_{xx} = \cos(U)$

c. $U_t - UU_{xx} + 12xU_x = U$.

d. $Y_{xxx} - \cos Y = Y_t$.

e. $V_{xt} - \sin V = \exp(x+t)$.

f. $Y_{xx} + \cos(xy)Y_{yxy} = Y + \ln(x^2 + y^3)$.

g. $U_t = U_{xx} - 12\, U\, U_x$.

h. $V_{yx} + V_x + V_y = V_{xyy}$.

i. $U_{tt} - \cos(U_x) = U$.

j. $\cos x \cdot U_x + \sin t \cdot U_t = U$.

k. Schrodinger equation (with potential):

$$-i\partial_t \psi = \frac{1}{2m}\nabla^2\psi + V(x)\psi. \tag{2.37}$$

l. Monge–Ampere equation (2 variable):

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y). \tag{2.38}$$

m. Monge–Ampere equation (multi-variable):

$$\det\left[\frac{\partial^2 u}{\partial x^i\, \partial x^j}\right] = f\left(x^i, u, \frac{\partial u}{\partial x^i}\right). \tag{2.39}$$

n. Navier–Stokes equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}. \tag{2.40}$$

o. Tricomi equation:

$$y\, U_{xx} + U_{yy} = 0. \tag{2.41}$$

p. Frobenius–Mayer equation (special case, one dependent variable):

$$\frac{\partial U}{\partial x^i} = F_i(x, U). \tag{2.42}$$

(More on this PDE below.)

q. Biharmonic equation:

$$\nabla^4 \Psi = 0. \tag{2.43}$$

That is, $(\nabla^2)^2\ \Psi = 0$, or more explicitly:

$$\left[\partial_x^2 + \partial_y^2 + \partial_z^2\right]^2 \Psi = 0. \tag{2.44}$$

r. Benjamin–Bona–Mahony equation:

$$u_t + u_x + u u_x - u_{xxt} = 0. \tag{2.45}$$

s. Chaplygin equation:

$$u_{xx} + \frac{c^2\ y^2}{c^2 - y^2}\ u_{yy} + y\ u_y = 0. \tag{2.46}$$

t. Boissinesq equation:

$$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}. \tag{2.47}$$

u. Euler–Darboux equation:

$$u_{xy} + \frac{\alpha\ u_x - \beta\ u_y}{x - y} = 0. \tag{2.48}$$

v. Korteweg–deVries–Burger equation:

$$u_t + 2uu_x - \nu\, u_{xx} + \mu\, u_{xxx} = 0. \tag{2.49}$$

w. Kirchever–Novikov equation:

$$\frac{u_t}{u_x} = \frac{1}{4}\frac{u_{xxx}}{u_x} - \frac{3}{8}\frac{u_{xx}^2}{u_x^2} + \frac{3}{8}\frac{4u^3 - g_2 u - g_3}{u_x^2}. \tag{2.50}$$

(Start by simplifying this a little.)

x. Lin–Tsien equation:

$$2u_{tx} + u_x\, u_{xx} - u_{yy} = 0. \tag{2.51}$$

y. Monge–Ampere equation (generalized):

$$\begin{aligned} &E(x,y,U,U_x,U_y)\left[U_{xx}U_{yy} - U_{xy}^2\right] \\ &+A(x,y,U,U_x,U_y)\,U_{xx} + B(x,y,U,U_x,U_y)\,U_{xy} + C(x,y,U,U_x,U_y)\,U_{yy} \\ &\qquad +D(x,y,U,U_x,U_y) = 0, \end{aligned} \tag{2.52}$$

or even more generally (the multi-variable case):

$$E(x^i,U,\partial_i U)\ \det\left[\frac{\partial^2 U}{\partial x^i\,\partial x^j}\right] + \sum_{ij} A^{ij}(x^i,U,\partial_i U)\,U_{,ij} + D(x^i,U,\partial_i U) = 0. \tag{2.53}$$

z. Cauchy–Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \tag{2.54}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.55}$$

Iterate these Cauchy–Riemann equations to find a pair of PDEs that decouple — they depend only on $u$, and only on $v$, but not both.

# Chapter 3

# General solutions

Unlike ODEs, the notion of a *general solution* of a PDE can get extremely complicated, very quickly.

## 3.1 Definition

In these lectures, when the term "general solution" is used, it will be meant in the following rather special sense:

**Definition 6 General solution:**

*A solution $U(x, y)$ of an $n$-th order PDE with a single dependent variable*

$$F\left(x, y, U(x, y), U^{(1)}, U^{(2)}, ., ., ., U^{(n)}\right) = 0 \tag{3.1}$$

*is a "general solution" if $U$ depends on $n$ arbitrary independent functions.*

**Warning 1** *Note "independent functions"* not *"independent constants".*

This is a direct extension of the notion of a general solution taken from the case of ODEs:

- Recall that for an ODE, a general solution is a solution depending on $n$ independent constants: and recall that we arrived at this idea by noting that, in principle, to solve an $n$-th order DE, we essentially need to integrate $n$ times — and each integration introduces an arbitrary constant. The same applies of course to a PDE — to solve it, we in principle must integrate $n$ times, and each integration introduces a function (rather than a constant). The examples below illustrate this fact.
- When it comes to a general PDE, or general systems of PDEs, the precise situation regarding a general solution can only be clearly stated using the relatively sophisticated work of Riquier and Janet, (brief comments in the next chapter). It is not appropriate to describe this Riquier–Janet formalism in MATH 301.

**Reminder 1**
*Even for ODEs, in the* nonlinear *case life is a lot more complicated than you might at first suspect.*

## 3.2 Examples

Here are some simple examples of "general solutions":

1. The equation
$$\frac{\partial U}{\partial x} = 0. \tag{3.2}$$
Keep in mind what the partial derivative means — you are differentiating $U$ with respect to $x$, treating $y$ as if it were constant. To regain $U$, set:
$$U(x, y) = G(y) \tag{3.3}$$
where $G$ is an arbitrary "constant", which, since $y$ is considered to be an independent constant, is allowed to be a *different* arbitrary "constant" for each specific value of $y$. That is $G(y)$ is generally a *function* of $y$.

   You should then check this final result by differentiating.

   Note that for the general solution of this particular $1^{st}$ order PDE there is manifestly *one* arbitrary function $G(y)$.

2. The equation

$$\frac{\partial U}{\partial x} = f(x,y), \qquad \text{for some given } f(x,y). \tag{3.4}$$

Keep in mind what the partial derivative means — you are differentiating $U$ with respect to $x$, treating $y$ as if it were constant. To regain $U$, then it would seem that we should integrate with respect to $x$, again keeping $y$ constant:

$$U(x,y) = \int_{y \text{ constant}} f(x,y)\ \mathrm{d}x + G(y) \tag{3.5}$$

where $G$ is an arbitrary "constant", which, since $y$ is considered constant, is allowed to be a *different* arbitrary "constant" for each specific value of $y$. That is $G(y)$ is generally a *function* of $y$.

Introducing the dummy variable $\bar{x}$ we can make this general solution more explicit as:

$$U(x,y) = \int_{x_0}^{x} f(\bar{x},y)\ \mathrm{d}\bar{x} + G(y). \tag{3.6}$$

You should then check this final result by differentiating.

Note that for the general solution of this particular $1^{st}$ order PDE there is manifestly *one* arbitrary function $G(y)$.

3. The equation

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = g(x,y), \tag{3.7}$$

where $g$ is a given function.

Here it will pay to change the independent variables, to new ones $s$, $t$ defined by

$$s = x + y; \qquad t = x - y. \tag{3.8}$$

So that

$$x = \frac{s+t}{2}; \qquad y = \frac{s-t}{2}. \tag{3.9}$$

But by the chain rule

$$\frac{\partial}{\partial x} = \frac{\partial s}{\partial x}\frac{\partial}{\partial s} + \frac{\partial t}{\partial x}\frac{\partial}{\partial t}. \tag{3.10}$$

Then it is easy to show that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}; \tag{3.11}$$

and similarly

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s} - \frac{\partial}{\partial t}; \tag{3.12}$$

or equivalently

$$\frac{\partial}{\partial s} = \frac{1}{2}\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right\}; \tag{3.13}$$

$$\frac{\partial}{\partial t} = \frac{1}{2}\left\{\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right\}. \tag{3.14}$$

Hence the original PDE is

$$\frac{\partial U}{\partial s} = \frac{1}{2}\, g\left(\frac{s+t}{2}, \frac{s-t}{2}\right) = G(s,t), \tag{3.15}$$

which can now be solved in general as in the first example.

Doing so yields

$$U(s,t) = \int_{t \text{ constant}} G(s,t)\ \mathrm{d}s + H(t), \tag{3.16}$$

which we first re-write (explicitly using the dummy variable $\bar{s}$) as

$$U(s,t) = \frac{1}{2}\int_{s_0}^{s} g\left(\frac{\bar{s}+t}{2}, \frac{\bar{s}-t}{2}\right)\ \mathrm{d}\bar{s} + H(t). \tag{3.17}$$

Now follow this by a change of independent variables back to $x$ and $y$ to produce our final answer:

$$U(x,y) = \frac{1}{2}\int_{s_0}^{x+y} g\left(\frac{\bar{s}+[x-y]}{2}, \frac{\bar{s}-[x-y]}{2}\right)\ \mathrm{d}\bar{s} + H(x-y). \tag{3.18}$$

You should then check this final result by differentiating.

Note that for the general solution of this particular $1^{st}$ order PDE there is manifestly *one* arbitrary function $H(x-y)$.

4. The equation
$$\frac{\partial^2 U}{\partial x\,\partial y} = H(x,y), \tag{3.19}$$
for a given function $H(x,y)$.

Take the LHS to be
$$\frac{\partial}{\partial x}\left[\frac{\partial U}{\partial y}\right], \tag{3.20}$$
and proceed as in the first example, integrating with respect to $x$, treating $y$ as constant:
$$\frac{\partial U}{\partial y} = \int_{y\ \text{constant}} H(x,y)\ \mathrm{d}x + g(y). \tag{3.21}$$
where $g$ is an arbitrary function.

Now integrate with respect to $y$, treating $x$ as a constant:
$$U(x,y) = \int_{x\ \text{constant}} \left[\int_{y\ \text{constant}} H(x,y)\ \mathrm{d}x\right] \mathrm{d}y + G(y) + F(x) \tag{3.22}$$
where $F$ is another arbitrary function, and $G$ is the integral of $g$ (and so is an arbitrary function).

In terms of dummy variables $\bar{x}$ and $\bar{y}$ our general solution can be rewritten in the explicit form:
$$U(x,y) = \int_{y_0}^{y} \left[\int_{x_0}^{x} H(\bar{x},\bar{y})\ \mathrm{d}\bar{x}\right] \mathrm{d}\bar{y} + G(y) + F(x). \tag{3.23}$$
You should then check this final result by differentiating.

Note that for the general solution of this particular $2^{nd}$ order PDE there are manifestly *two* arbitrary functions $G(y)$ and $F(x)$.

5. The equation
$$\frac{\partial^2 U}{(\partial x)^2} = H(x,y), \tag{3.24}$$
for a given function $H$.

Note
$$\frac{\partial^2 U}{(\partial x)^2} = \frac{\partial}{\partial x}\left[\frac{\partial U}{\partial x}\right]. \tag{3.25}$$

Proceeding as before, integrating [twice] with respect to $x$ and keeping $y$ fixed, we find

$$U(x,y) = \int_{y \text{ constant}} \left[ \int_{y \text{ constant}} H(x,y)\, \mathrm{d}x \right] \mathrm{d}x + x\, G(y) + F(y), \quad (3.26)$$

where $G$ and $F$ are arbitrary "constants"; but potentially *different* constants for each value of $y$.

In terms of dummy variables, now $\bar{x}$ and $\tilde{x}$, our general solution can be rewritten in the explicit form:

$$U(x,y) = \int_{x_0}^{x} \left[ \int_{x_0}^{\tilde{x}} H(\bar{x},y)\, \mathrm{d}\bar{x} \right] \mathrm{d}\tilde{x} + x\, G(y) + F(y). \quad (3.27)$$

You should then check this final result by differentiating.

Note that for the general solution of this particular $2^{nd}$ order PDE there are manifestly *two* arbitrary functions $G(y)$ and $F(y)$.

From these four examples the general pattern should be obvious.

**Comment 1** *Consider the general change of independent variables (that is, the general two-dimensional change of coordinates):*

$$(x,y) \to (u,v) = (u(x,y), v(x,y)) \quad (3.28)$$

*What happens to the partial derivatives? The general rule is this:*

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}; \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}. \quad (3.29)$$

*This should be obvious — think of it as an application of the chain rule. (That is, the multi-variable chain rule...)*

*Similarly if we consider the inverse transformation*

$$(u,v) \to (x,y) = (x(u,v), y(u,v)) \quad (3.30)$$

*we see*

$$\frac{\partial}{\partial u} = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y}; \qquad \frac{\partial}{\partial v} = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y}. \quad (3.31)$$

**Comment 2** *You should also be prepared for notation such as*

$$(x, y) \to (u, v) = (u(x, y), v(x, y)) \tag{3.32}$$

$$\partial_x = (\partial_x u)\, \partial_u + (\partial_x v)\, \partial_v; \qquad \partial_y = (\partial_y u)\, \partial_u + (\partial_y v)\, \partial_v. \tag{3.33}$$

*and*

$$(u, v) \to (x, y) = (x(u, v), y(u, v)) \tag{3.34}$$

$$\partial_u = (\partial_u x)\, \partial_x + (\partial_u y)\, \partial_y; \qquad \partial_v = (\partial_v x)\, \partial_x + (\partial_v y)\, \partial_y. \tag{3.35}$$

## 3.3 Exercises

**Reminder:**

- The general solution to an ODE of the $n$-th order contains $n$ arbitrary and independent *constants*. For PDEs the situation is much more complicated, but nevertheless we will define a general solution of a single PDE of the $n$-th order in a *single* unknown $U$ as a solution involving *$n$ arbitrary functions*. This of course is not the best definition, but it will do here.
- In the case of an ODE the general solution completely defines its corresponding ODE in the sense that, given a function depending on $n$ independent and arbitrary constants, there should only be one $n$-th order ODE which has that function as its general solution [to see this, recall that we considered an ODE as a means of encoding all the derivatives of its solution, the $n$ arbitrary constants being the first few derivatives, at $x = 0$ say, that are not defined by the ODE].
- In a similar fashion, given a function $u(x, y)$ which also involves $n$ independent functions, there will be a (hopefully unique) PDE of $n$-th order that will have that function as its general solution. One of the questions below asks you to find the corresponding PDE for given general solutions.

### 3.3.1 From general solution to PDE

Consider

$$u = f(x - y). \tag{3.36}$$

Then

$$\frac{\partial u}{\partial x} = f'(x - y); \qquad \frac{\partial u}{\partial y} = -f'(x - y). \tag{3.37}$$

Eliminate $f'$, obtaining

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{3.38}$$

This PDE now makes no reference to $f$, and the general solution of this PDE is the equation you started from.

- Using this technique, eliminate the arbitrary functions from the following and so obtain partial differential equations of which they are the general solution:

  a. $u = f(x + y)$.
  b. $u = g(xy)$.
  c. $u = f(x + y) + g(x - y)$.
  d. $u = x^n\, h(y/x)$.
  e. $v = g(x^2 + y^2)$.
  f. $v = f(x^2 - y^2)$.
  h. $v = f(x^2 - y^2) + g(x^2 + y^2)$.
  i. $v = h(2x - y) - g(2x + y)$.

- Now consider a general solution specified by the system of two equations:

  $$u(x, y) = \alpha(x, y)\, x + w(\alpha(x, y))\, y + v(\alpha(x, y));$$
  $$0 = x + w'(\alpha(x, y))\, y + v'(\alpha(x, y)).$$

  Eliminate the arbitrary functions $w(\alpha)$ and $v(\alpha)$, and the parameter $\alpha$ itself, to obtain a PDE for $u(x, y)$.

  (You should find a particularly simple example of a Monge–Ampere equation.)

  *Hint:* See Courant and Hilbert — Volume 2 page 10.

- Suppose you are given a class of functions $y(x:\vec{a}) = f(x:a_1, a_2, \ldots, a_n)$ of the *single* variable $x$, where the class of functions is parameterized by $n$ arbitrary parameters $a_1$, $a_2$, …, $a_n$, denoted collectively by $\vec{a}$.

  Suppose further that the parameters come under the heading of being both "arbitrary and independent", namely, suppose that the following determinant is non-zero $(i, j = 1, \ldots, n)$:

$$\det\left[\left(\frac{\partial}{\partial x}\right)^i \frac{\partial}{\partial a_j}\, f(x:\vec{a})\right] \neq 0. \qquad (C) \tag{3.39}$$

  Then you can easily prove that $y(x:\vec{a})$ must be the general solution of *some* ODE of the $n$-th order. You do this effectively by eliminating the parameters $a_k$, $k = 1, 2, \ldots, n$.

  Consider the $n$ equations:

$$\begin{aligned} y &= y(x:a_1, a_2, \ldots, a_n) && (3.40)\\ y' &= f'(x:a_1, a_2, \ldots, a_n) && (3.41)\\ y'' &= f''(x:a_1, a_2, \ldots, a_n) && (3.42)\\ . &\;. \;\ldots && (3.43)\\ . &\;. \;\ldots && (3.44)\\ . &\;. \;\ldots && (3.45)\\ y^{(n-1)} &= f^{(n-1)}(x:a_1, a_2, \ldots, a_n) && (3.46) \end{aligned}$$

  These are $n$ equations relating the $n$ variables $y, y', y'', .., y^{(n-1)}$ to the $n$ parameters $a_1, a_2, \ldots, a_n$.

  But because of the condition $(C)$ above, the *inverse function theorem* guarantees that you can (at least locally) solve these equations, to thereby find the parameters $a_1, a_2, \ldots, a_n$ as functions of the variables $y, y', y'', \ldots, y^{(n-1)}$, and $x$:

$$a_k = A_k(x:y, y', y'', \ldots, y^{(n-1)}) \tag{3.47}$$

  for $k = 1, 2, ..., n$ and *some* functions $A_k$ of the indicated variables..

  Now use these functions $A_k$ to eliminate the variables $a_1, a_2, \ldots, a_n$ in the expression for the $n$-th derivative of $y$:

$$y^{(n)} = f^{(n)}(x:a_1, a_2, .., a_n) \tag{3.48}$$

in favour of the derivatives $y, y', y'', .., y^{(n-1)}$. That is

$$y^{(n)} = f^{(n)} \left( x : A_i(x : y, y', y'', ...y_{(n-1)}) \right). \tag{3.49}$$

In doing so, you will end up with a relation between the derivatives of the function $y$ of the form:

$$y^{(n)} = G(x, y', y'', ..., y^{(n-1)}), \tag{3.50}$$

which is an ODE in $y$ of order $n$.
(In fact it's even guaranteed to be quasi-linear).

*Can you now set up an analogous way of obtaining a PDE?*

Two examples:

- Specifically, consider the general class of functions

$$u = f(x, y; \alpha, \beta) \tag{3.51}$$

  By differentiating with respect to $\alpha$ and $\beta$, and then appealing to the inverse function theorem, argue that this general class of functions is the solution set of the generic first-order PDE

$$F(x, y, u, u_x, u_y) = 0. \tag{3.52}$$

- What happens for the three-parameter general class of functions

$$u = f(x, y; \alpha, \beta, \gamma)? \tag{3.53}$$

Develop a general formalism for going from a parameterized class of "solutions" to the PDE that "generates" that solution class.

(When all else fails, look up Courant and Hilbert, volume 2, pp. 8 ff. for some hints...)

### 3.3.2 From PDE to general solution

By integrating out the partial derivatives in the following PDEs, find the general solution.

a. $U_{xy} = y\, U_x^3$.

b. $U_{xy} = xy\, U_y$.

c. $U_{xy} = y\, U_y + x^3 y^2$.

d. $U_{xx} = y\, U_x + xy$.

e. $U_x = U_y$.

f. $\alpha\, U_x + \beta\, U_y = 0$. (Treat $\alpha$ and $\beta$ as given constants.)

g. $U_x\, g_y(x, y) - U_y\, g_x(x, y) = 0$. (Treat $g(x, y)$ as given.)

h. $U_{xxyy} = 0$.

This exercise illustrates the rather complex way that the arbitrary functions could appear in the general solution.

Now try to find the general solutions for

i. $\alpha(U)\, U_x - \beta(U)\, U_y = 0$.

j. $U_x\, g_y(x, y, U) - U_y\, g_x(x, y, U) = 0$. (Treat $g(x, y, U)$ as given.)

In these cases you will have to be satisfied with an *implicit* relation for $U(x, y)$ in terms of some arbitrary function.

Finally:

k. Hence or otherwise show that the general solution of the (1+1) PDE

$$v_t + v v_x = 0 \tag{3.54}$$

is given *implicitly* by

$$v(t, x) = f\Big(x - v(t, x) t\Big). \tag{3.55}$$

(**Challenge**: Try to come up with a physical model for a situation where this PDE is relevant.)

l. Hence or otherwise show that the general solution of the (3+1) PDE

$$\vec{v}_t + (\vec{v} \cdot \nabla)\, \vec{v} = 0 \tag{3.56}$$

is given *implicitly* by

$$\vec{v}(t, x) = \vec{f}\Big(\vec{x} - \vec{v}(t, \vec{x})t\Big). \tag{3.57}$$

(**Challenge**: Try to come up with a physical model for a situation where this PDE is relevant.)

### 3.3.3 General solution versus singular solution

The definition of general solution for a single first order PDE in a single unknown was that it be a solution involving one arbitrary function.
As for ODEs, the general solution may not always cover all possible solutions, (those exceptional solutions are called singular solutions).
See, for example, Courant and Hilbert, volume 2 pp. 2 ff. (§1).
Here is an example:
Consider the (1+1) dimensional PDE

$$\frac{\partial U}{\partial x} - \frac{\partial U}{\partial y} = 2\sqrt{U}. \tag{3.58}$$

i. Explicitly verify that $U = [x + \eta(x + y)]^2$ is a solution, for any arbitrary function $\eta(\_)$.

   Therefore, since we have a solution to a first order PDE containing one arbitrary function, this is an example of a "general solution".

ii. Show that $U = 0$ is also a specific solution to the equation.

iii. Show that one cannot express the specific solution $U = 0$ in the form $[x + \eta(x + y)]^2 = 0$ for *any* function $\eta$.

   Thus we have found a specific solution that does not follow from the general solution!!

*For a general discussion of singular solutions for such equations see*
M. J. Hill, Proceedings of the London Mathematical Society, 1917.

**Hint:** Using the substitution $U = W^2$ show that the PDE reduces to

$$W\left(\frac{\partial W}{\partial x} - \frac{\partial W}{\partial y} - W\right) = 0. \qquad (3.59)$$

This should make it obvious why there are two disconnected branches of solutions.

### 3.3.4 General solutions

Write down, using whatever technique you find easiest, the general solution for these PDEs:

a. $y\,\dfrac{\partial U}{\partial x} - x\,\dfrac{\partial U}{\partial y} = 0.$

b. $x\,\dfrac{\partial U}{\partial x} + y\,\dfrac{\partial U}{\partial y} = 0.$

c. $x\,U\,\dfrac{\partial U}{\partial x} + y\,U\,\dfrac{\partial U}{\partial y} = xy.$

d. $\tan x\,\dfrac{\partial U}{\partial x} + \tan y\,\dfrac{\partial U}{\partial y} = \tan U.$

e. $y\,\dfrac{\partial U}{\partial x} + z\,\dfrac{\partial U}{\partial y} - x\,\dfrac{\partial U}{\partial z} = 0.$

### 3.3.5 Boundary value problems

Solve the following boundary value problems by first obtaining, using that innate cunning for which Math 301 students are renowned, the general solutions of the PDEs and then fitting them to the given boundary conditions:

a. $U_{xx} = \frac{1}{c^2}\,U_{tt}$, given that $U(x,0) = 0$ and $U_t(x,0) = 1/(1+x^2)$.

b. $U_{xx} = 2xy$, given that $U(0,y) = y^2$ and $U_x(0,y) = y$.

c. $V_{xy} = 1$, given that $V = 0$ and $V_x = 0$ when $x + y = 0$.

Classify these BC as to whether they are Dirichlet, Neumann, Robin, or something else.

# Chapter 4

# Existence and uniqueness

## 4.1 Definition: Solution of a PDE

**Definition 7 Solution of a PDE:**

*A function $U = U(x,y)$ is a solution of the PDE*

$$F\left(x, y, U, U^{(1)}, U^{(2)}, \ldots, U^{(n)}\right) = 0 \tag{4.1}$$

*on a region $W$ of the plane $\mathbb{R}^2$ if:*

- *Both $U(x,y)$ and its partial derivatives*

$$U^{(1)}(x,y), \ldots, U^{(n)}(x,y) \tag{4.2}$$

  *exist on $W$.*

- *For every $(x,y)$ in $W$*

$$F\left(x, y, U(x,y), U^{(1)}, U^{(2)}, \ldots, U^{(n)}\right) = 0. \tag{4.3}$$

*That is, the function $U$ can be differentiated as often as necessary, and when substituted back into the PDE it makes the equation true.*

**Warning 2**
*Sometimes solutions in the sense given above are called "classical solutions". (Sometimes they are called "strong solutions".)*

**Warning 3**
*There is a whole separate issue of so-called "weak solutions" of PDEs. Not appropriate for this part of MATH 301. (Though you could always look up Chapter 10 in Olver.)*

| |
|---|
| The general situation regarding existence and uniqueness of solutions for systems of PDEs is considerably more complicated than for ODEs. <br><br> Below we give a very cursory description of the situation. |

## 4.2 The Cauchy Theorem

Only in the case where all functions involved in defining the PDE are *analytic* is there an existence and uniqueness result of complete generality resembling the EUS (Existence and Uniqueness of Solutions) theorem for ODEs.

The most basic of the EUS theorems, which is easy to state and to understand, and which initiated many of the later developments in the theory of PDEs, is due to Cauchy.

See, for example, Courant and Hilbert, volume 2 pp. 39 ff. (§7).

**Reminder 2**
*Analytic, $C^\omega$, means infinitely differentiable and expandable as a power series with non-zero radius of convergence.*
*Smooth, $C^\infty$, just means infinitely differentiable.*
*$C^2$ means twice differentiable [with continuous derivative].*
*$C^1$ means once differentiable [with continuous derivative].*
*$C^0$ means continuous.*

**Example 1**
$\exp(x)$ *is* $C^\omega$ *for finite* $x$.
$\exp(1/x)$ *is* $C^\infty$ *for finite* $x$, *but not even* $C^0$ *at* $x = 0$.
$|x^3|$ *is* $C^2$ *but not* $C^3$.
$|x|$ *is* $C^0$ *but not* $C^1$.

**Theorem 1 Cauchy**

*Consider the PDE*

$$\frac{\partial U}{\partial x} = f\left(x, y, U, \frac{\partial U}{\partial y}\right). \tag{4.4}$$

*This is a specific example of a first-order PDE in one dependent variable and two independent variables.*

*Consider the initial condition that*

$$U(0, y) = g(y), \tag{4.5}$$

*is, at $x = 0$, a prescribed analytic function of the independent variable $y$.*

*Suppose furthermore that $f(\bullet, \bullet, \bullet, \bullet)$ is an analytic function of its arguments.*

*Then there exists one, and only one, unique solution satisfying these initial conditions.*

**Notes:**

- Note that you have to make some extremely powerful assumptions to be able to derive the theorem — much more powerful than those needed for the EUS (existence and uniqueness theorem) for ODEs.
- You can find a generalized version of the theorem and proof discussed fully in Courant and Hilbert (reference below), [volume 2] pages 39–56.
- Note that you are only trying to solve a first-order PDE, but to derive the theorem you need to make *analyticity* assumptions for $f(x, y, \cdots)$. That is — infinitely differentiable *and* a convergent Taylor series.
- So the hypotheses you have to put in are very strong compared to the result you wish to prove.
- Note that the two independent variables $x$ and $y$ are treated asymmetrically.

- Cauchy's theorem can be generalized in a number of ways:
  - To many independent variables, to higher order PDEs, and to systems of PDEs. This is relatively "straightforward" and leads to the Cauchy–Kowalewsky theorem.
  - To more complicated though analytic PDEs — this leads to the Riquier–Janet theory.
    (Not even Olver wants to open that particular can of worms...)
    (While Google can sometimes be your friend, in this particular case it will lead you into a rabbit warren...)
  - To many different non-analytic but relatively simple PDEs — these are often the most useful EUS theorems in practice.

## 4.3 The Cauchy–Kowalewsky Theorem

A reasonably well-known generalization of the Cauchy theorem (which is however still a very special case of the Riquier–Janet theory, which I will not even try to discuss) is the Cauchy–Kowalewsky Theorem, which I quote below for the case of a system of PDEs of the $k$-th order with several dependent variables $U^A$, which are functions of the $n+1$ independent variables $x, y^1, y^2, \dots, y^n$.

Note that *one* of the independent variables, $x$, has been singled out for special treatment!

(That is, one of the coordinates is treated differently from the others!)

**Historical note 1** *Since the Russian alphabet is radically different from both the English alphabet, and since she published a lot of work in German and French, [and Swedish?], poor Sophie (Sofia, Sonya) Kowalewsky's name has gotten rather mangled over the years. In addition to Kowalewsky I have seen Kovalevskaya, Kowalevskaya, and Kovalevski. I'm sure there's other variants out there.*

*See:* http://en.wikipedia.org/wiki/Sofia_Kovalevskaya

**Historical note 2** *Sophie Kowalewsky (1850-91) did important work in partial differential equations. Born in Moscow, she married a paleontologist and moved to Germany. At the University of Heidelberg she studied privately with the great mathematician Weierstrass; women were not allowed at lectures. She received a degree in absentia in 1874 for her thesis on partial differential equations. Her most famous work tells conditions when a partial differential equation has a solution that is unique and analytic.*

*She won the Paris Academy Prize in 1888 for a paper on the integration of the equations of motion for a solid body rotating around a fixed point; the paper was of such high quality that the announced award money was doubled. In 1889 she became a professor of mathematics at Stockholm. In addition to her work in mathematics, she wrote some noted novels depicting life in Russia.*

**Historical note 3** *Courant and Hilbert credit Cauchy with the basic idea for this theorem, and credit Kowalewsky with carrying out the proof "in a rather general manner".*

See, for example, Courant and Hilbert, volume 2 pp. 39 ff. (§7).

**Theorem 2 Cauchy–Kowalewsky**

*Consider the system of PDEs*

$$\frac{\partial^k U^A}{(\partial x)^k} = f^A\left(x, y^1, ..., y^n, U^B, \frac{\partial U^B}{\partial x}, \dots, \frac{\partial^{k-1} U^B}{(\partial x)^{k-1}}, \frac{\partial U^B}{\partial y^i}, \dots, \frac{\partial^k U^B}{(\partial y^i)^k}\right). \quad (4.6)$$

*Consider the initial conditions that the functions*

$$U^A(0, y^1, \dots, y^n), \quad \frac{\partial U^A}{\partial x}(0, y^1, \dots, y^n), \quad \frac{\partial^2 U^A}{(\partial x)^2}(0, y^1, \dots, y^n), \quad (4.7)$$

*and*

$$\frac{\partial^{k-1} U^A}{(\partial x)^{k-1}}(0, y^1, \dots, y^n) \quad (4.8)$$

*are, at $x = 0$, all prescribed analytic functions of the independent variables $y^1$, …, $y^n$.*

*Suppose furthermore that the functions $f^A(\bullet, \bullet, \bullet, \cdots)$ are analytic functions of their arguments.*

*Then there exists one and only one unique solution satisfying these initial conditions.*

- When the PDE is presented in this particular manner it is said to be in "normal form".
- Note that this is a $k$'th order system of PDEs in $(n + 1)$ independent variables — that is, defined on a space with $(n + 1)$ coordinates.
- The number of equations, and hence the number of dependent variables, is arbitrary.
- Note that the initial conditions are all specified on the very special hyperplane $x = 0$.
- You can find the theorem and proof discussed fully in Courant and Hilbert (reference below), [volume 2] pages 39–56.
- Note that the Courant and Hilbert book is definitely not light reading; it is however a gold-mine of highly technical information.
- Note that you are only trying to solve a $k$'th order system of PDEs, but to derive the theorem you need to make *analyticity* assumptions for $f(x, \cdots)$. That is — infinitely differentiable *and* a convergent Taylor series. The hypotheses you have to put in are extremely strong compared to the result you wish to prove.
- To see what is going on it is convenient to work with *systems* of first-order PDEs in *two* independent variables $x$ and $y$. As Courant and Hilbert say, "there is no modification necessary for more independent variables". Because we are now dealing with *systems* of first-order PDEs, this is still a significant generalization of the original Cauchy theorem.

**Theorem 3 Cauchy–Kowalewsky (simplified 2 dimensions)**

*Consider the system of PDEs*

$$\frac{\partial U^A}{\partial x} = f^A \left( x, y, U^B, \frac{\partial U^B}{\partial y} \right). \tag{4.9}$$

*Consider the initial conditions that the fields*

$$U^A(0, y) = g^A(y), \tag{4.10}$$

*are, at $x = 0$, all prescribed analytic functions, $g^A(y)$, of the independent variable $y$.*

*Suppose, furthermore, that the $f^A(\bullet, \bullet, \bullet, \bullet)$ are analytic functions of their arguments.*

*Then there exists one, and only one, unique solution satisfying these initial conditions.*

- Courant and Hilbert state:
  "To prove the theorem one first formally constructs power series for the solution and then shows the uniform convergence of these series."
- The details are "straightforward" and are left as an exercise for the reader.
- Remember how to translate that code word "straightforward"?

## 4.4 The non-analytic case

If the PDE involves non-analytic coefficients, or some non-analytic function $F(\_\_, \_\_, \_\_)$ relating the various partial derivatives, then the general situation is not particularly general at all:

- A single first-order PDE in a single unknown, with given IC, is known to have a unique solution, and methods for its construction are available.
  - That is, equations of the form
    $$F\left(x, y, U^{(1)}, U\right) = 0 \tag{4.11}$$
    are sufficiently simple that EUS theorems can be developed.
  - See, for example, Courant and Hilbert, volume 2 pp. 22 ff. (§4).
- We can also develop rather simpler EUS theorems for first-order *linear* equations of the form
  $$\sum_{i=1}^{n} a^i(x_1, \ldots, x_n) \, \frac{\partial U(x_1, \ldots, x_n)}{\partial x^i} \tag{4.12}$$
  $$+b(x_1, \ldots, x_n) \, U(x_1, \ldots, x_n) + f(x_1, \ldots, x_n) = 0. \tag{4.13}$$
  Such an equation can be directly related to a system of first-order ordinary DEs, leading to the theory of "characteristics".

  [See Forsyth, (reference below), Courant and Hilbert, Hormander, (reference below), for more details.]
  - See, for example, Courant and Hilbert, volume 2 pp. 28 ff. (§5).
- Similarly we can also develop rather simple EUS theorems for *some* first-order *quasi-linear* equations of the form
  $$\sum_{i=1}^{n} a^i(x_1, \ldots, x_n, U) \, \frac{\partial U(x_1, \ldots, x_n)}{\partial x^i} + f(x_1, \ldots, x_n, U) = 0. \tag{4.14}$$
  This leads to a generalization of the theory of characteristics.
  - See, for example, Courant and Hilbert, volume 2 pp. 28 ff. (§5).

- For a *system* of first-order equations in a single unknown, consistency conditions can be formulated, and methods for the construction of the unique solution for given consistent initial conditions have been found — see, for example, Forsyth again.

  (This can be transformed into a special case of the Frobenius–Mayer system, as will be discussed below).
- A general system of first order PDEs in many unknowns is very difficult to analyse, and only special cases are known (see, for example, Forsyth again).
- Warning: You can always take a *single* $n$'th-order PDE, in *one* dependent variable, and recast it as a *system* of $n$ first-order PDEs, in $n$ dependent variables.

  However the converse is *not* true for PDEs (though it is true for ODEs). That is:

  - Given a system of $n$ first-order ODEs it is in general possible to reduce this to a single equivalent $n$'th order ODE.
  - Given a system of $n$ first-order PDEs it is in general *not* possible to reduce this to a single equivalent $n$'th order PDE.
  - See, for example, Courant and Hilbert, volume 2 pp. 58 ff. (Appendix 2 to Chapter 1).
- There is no single unified theory of PDEs — it's very much a collection of special cases (some more general than others).

**References:**


- Hormander, L., *Linear Partial Differential Equations*,
  Academic Press N.Y. 1963.

- Courant R., and D. Hilbert,
  *Methods of Mathematical Physics Vols 1 and 2*, Interscience 1966.

- Forsyth R., *Differential Equations*, in six volumes,
  Oxford University Press, (1906 onwards).

  This opus covers a large number of techniques, many of which are now mostly forgotten, but which crop up from time to time in research papers.

## 4.5 EUS results for specific PDEs

Although, as we have just seen, the *general* theory of EUS for generic PDEs is quite patchy and relatively ill-developed (compared to EUS for ODEs), the situation for *specific* PDEs is often (not always) a lot better. If some specific PDE has become important for some specific physical/ chemical/ biological/ financial/ military or other reason, then there has generally been a lot of hard work done on the EUS problem for that specific PDE. So in some specific cases we can say a lot, in other cases things are still a bit of a mess.

## 4.6 Exercise

Solve the following first order linear PDE:

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} = x \cos(xy). \tag{4.15}$$

Do this by making a cunning transformation of variables $s = x + y$, $t = x - y$ and rewriting the equation in terms of these variables.

**Challenge:** Read and understand the theory of characteristic curves.

**Challenge:** Read and understand the proof of the Cauchy theorem.

**Challenge:** Read and understand some advanced books on PDEs.

**Challenge:** Find, read, and understand some recent PhD theses on PDEs.

I reiterate:

- PhD theses are still being written on (advanced) first-order *systems* of PDEs.
- PhD theses are still being written on (advanced) second-order PDEs.
- PhD theses are still being written on the general theory of PDEs.

# Chapter 5

# Some $1^{st}$-order PDEs

## 5.1 Frobenius–Mayer systems

Frobenius–Mayer systems are a specific example of a system of PDEs that is sufficiently simple to enable us to obtain a EUS theorem *without having to make analyticity assumptions.*

### 5.1.1 Definition

**Definition 8 Frobenius/Mayer system:**

*One special case that is very important is the* **Frobenius** *or* **Mayer** *system*

$$\frac{\partial U^A}{\partial x^i} = F^A{}_i(x^1, \ldots, x^n, U^1, \ldots, U^m) \qquad (F) \qquad (5.1)$$

$$A = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, n \qquad (5.2)$$

*where the $m$ dependent variables $\{U^A\}$, (the "fields"), depend on the $n$ independent variables $\{x^i\}$, (the "position in some $n$-dimensional space").*

*All these equations are all of first order.*

*In such a system there are as many PDEs as there are first-order derivatives of the dependent functions (i.e., $nm$ of them)*

**Notes:**

- We see that the Frobenius–Mayer PDE systems are examples of first-order quasi-linear PDE systems.
- The superscripts now tell you *which* of the $U$'s you are dealing with; *not* the order of the derivative.
- The only derivatives occurring above are first-order on the LHS.

  (And they occur linearly with coefficient unity.)
- The RHS of the system does not involve *any* derivatives.
- Just because it's important does not mean it's easy to find any explicit discussion of this system.
- You can find a discussion in Volume 1 of Spivak, chapter 6.
  See especially pages 254–257.
  (The notation is slightly different).
- You can find a discussion in Volume 5 of Forsyth, chapter 4.
  See especially pages 100 ff.
  (The notation is, unfortunately, seriously archaic).

**References:**


- Courant R., and D. Hilbert,
  *Methods of Mathematical Physics Vols 1 and 2*, Interscience 1966.

- Forsyth R., *Differential Equations*, in six volumes,
  Oxford University Press, (1906 onwards).

- Spivak, M., *A comprehensive introduction to differential geometry*,
  in six volumes, (Publish or Perish, Berkeley, 1979).

### 5.1.2 Integrability theorem

**Theorem 4 The Frobenius Complete Integrability Theorem:**

*Suppose the functions $F^A{}_i(\bullet, \dots)$ are $C^1$ functions of all their variables in a neighbourhood of the origin, for $A = 1, 2, \dots, m$ and $i = 1, 2, \dots, n$.*

*Then the Frobenius system (F) has a unique solution satisfying the IC*

$$U^A(0, 0, \dots, 0) = b^A \qquad (A = 1, 2, \dots, m) \tag{5.3}$$

*for arbitrary given $b^A$, if and only if*

$$\frac{\partial F^A{}_i}{\partial x^j} + \sum_{B=1}^{m} F^B{}_j \frac{\partial F^A{}_i}{\partial U^B} = \frac{\partial F^A{}_j}{\partial x^i} + \sum_{B=1}^{m} F^B{}_i \frac{\partial F^A{}_j}{\partial U^B} \qquad (C) \tag{5.4}$$

*for all $i$, $j$, and $A$ in their respective ranges.*

- Note that we only require $F$ to be $C^1$ instead of $C^\omega$. That $C^1$ is a *necessary* condition is obvious — it is required so that the relevant derivatives in the compatibility condition ($C$) exist.
- This Frobenius integrability theorem is an extremely important result. The condition ($C$) is effectively the requirement that the second partial derivatives should all commute:

$$\frac{\partial^2 U^A}{\partial x^i\, \partial x^j} = \frac{\partial^2 U^A}{\partial x^j\, \partial x^i}. \tag{5.5}$$

- To see *necessity* (not *sufficiency*) note that if the PDE defining the Frobenius–Mayer system is satisfied, then

$$\frac{\partial^2 U^A}{\partial x^i\, \partial x^j} = \frac{\mathrm{d}}{\mathrm{d}x^i} F^A_j(x, U(x)). \tag{5.6}$$

  Then by applying the chain rule

$$\frac{\partial^2 U^A}{\partial x^i\, \partial x^j} = \frac{\partial}{\partial x^i} F^A_j + \sum_{B=1}^{m} \frac{\partial F^A_j}{\partial U^B}\, \frac{\partial U^B}{\partial x^i}. \tag{5.7}$$

Now use the Frobenius–Mayer PDE again, we see

$$\frac{\partial^2 U^A}{\partial x^i\,\partial x^j} = \frac{\partial}{\partial x^i} F^A_j + \sum_{B=1}^{m} \frac{\partial F^A_j}{\partial U^B}\, F^B_i. \tag{5.8}$$

But the LHS is symmetric under interchange $i \longleftrightarrow j$.

This leads to the consistency condition $(C)$.

- You can find a full proof [both *necessity* and *sufficiency*] in Volume 1 of Spivak, chapter 6, pages 254–257. Note that Spivak's notation is slightly different.
- You can get a feel for how important the Frobenius integrability theorem is from Spivak's comment:

> The Frobenius theorem (which represents everything we know about partial differential equations) was used in [ ...long list of topics... ].

(See Spivak, volume 5, page 1).

This should be balanced against his further comment:

> Now it's really rather laughable to call these things partial differential equations at all. True ... partial derivatives are involved, but we do not posit any relationship between *different* partial derivatives; this comes out quite clearly in the proof [of the integrability theorem] where the equations are reduced to ordinary differential equations.

**Proof of the Frobenius integrability theorem**:
Consider, in the specified coordinate chart, the "straight line" $x^i(t) = t\,x^i$ and on this "straight line" solve the ODE

$$\frac{\mathrm{d}U^A(t)}{\mathrm{d}t} = x^i\ F^A{}_i(t\,x^i; U^B(t)); \qquad U^A(0) = b^A. \tag{5.9}$$

Since this is simply an ODE, (albeit a non-autonomous coupled ODE in $m$ variables), it will have unique solutions, at least on some finite interval. Now use the $U^A(t)$ to define $U^A(x^i)$ as follows

$$U^A(x^i) = b^A + \int_0^1 x^i\ F^A{}_i(t\,x^i; U^A(t))\ \mathrm{d}t. \tag{5.10}$$

These $U^A(x^i)$ certainly exist, but what PDEs do they satisfy? Let us compute

$$\begin{aligned}\partial_i U^a(x) &= \int_0^1 F^A{}_i(t\,x;U(t))\;\mathrm{d}t \\ &\quad +x^j\int_0^1 t\{\partial_i F^A{}_j(t\,x;U(t)) + \partial_B F^A{}_j(t\,x;U(t))\;\partial_i U^B(t)\}\;\mathrm{d}t.\end{aligned} \tag{5.11}$$

But that first term can be integrated by parts as

$$\int_0^1 F^A{}_i(t\,x;U(t))\;\mathrm{d}t = [tF^A{}_i(t\,x;U(t))]_0^1 - \int t\frac{\mathrm{d}}{\mathrm{d}t}[F^A{}_i(t\,x;U(t))]\mathrm{d}t \tag{5.12}$$

$$= F^A{}_i(x;U(x)) - \int_0^1 t\left\{F^A{}_{i,j}(t\,x;U(t)x^j + \partial_B F^A{}_i \dot{U}^B\right\}\mathrm{d}t \tag{5.13}$$

$$= F^A{}_i(x;U(x)) - \int_0^1 t\left\{F^A{}_{i,j}(t\,x;U(t)x^j + \partial_B F^A{}_i F^B_j x^j\right\}\mathrm{d}t. \tag{5.14}$$

Combining

$$\partial_i U^a(x) = F^A{}_i(t\,x;U(x)) + \int_0^i tx^j\{F^A{}_{i,j} - F^A{}_{j,i} + \partial_B F^A_j \partial_i U^B - \partial_B F^A_i F^B_j\}\mathrm{d}t. \tag{5.15}$$

Now apply the consistency condition. Thence

$$\partial_i U^a(x) - F^A{}_i(t\,x;U(x)) = \int_0^1 t\,x^j\partial_B F^A_j\;\left\{\partial_i U^B(t) - F^B_i(tx,U(t))\right\}\mathrm{d}t. \tag{5.16}$$

This is an integral equation. One solution is clearly $\partial_i U^a(x) = F^A{}_i(t\,x;U(x))$. As long as the integral transform does not have eigenvalue unity, this will be the only solution. (And for sufficiently small $x$, where the integral transform is guaranteed to be small, this will certainly be the unique solution.) So under the stated consistency condition we have established the existence of a set of fields $U^A(x^i)$ such that $\partial_i U^a(x) = F^A{}_i(t\,x;U(x))$.

(This proof is slightly different from other presentations you might eventually track down, either on the internet or in various older texts. I feel the present discussion is pedagogically simpler.)
Various comments:

- Clearly if $n = 1$ [only one independent variable, one dimension] then condition $(C)$ is *always* satisfied. But this just means that if we have one independent variable then the 1-dimensional Frobenius equation

$$\frac{\partial U^A}{\partial x} = F^A(x, U^1, \dots, U^m) \qquad A = 1, 2, \dots, m, \qquad (1d\ F) \tag{5.17}$$

  is always integrable. This will be less of a surprise if we realise this is now an ODE, and change variables ($x \to t$, $U^A \to x^A$) to rewrite it in the more usual form

$$\frac{\mathrm{d}x^A}{\mathrm{d}t} = F^A(t, x^B) \qquad A = 1, 2, \dots, m. \tag{5.18}$$

  We already know, by elementary means, that this simple ODE is integrable.

- A second important case is $m = 1$ [only one *dependent* variable, one "field" but many dimensions] then condition $(C)$ reduces to

$$\frac{\partial F_i}{\partial x^j} + F_j \frac{\partial F_i}{\partial U} = \frac{\partial F_j}{\partial x^i} + F_i \frac{\partial F_j}{\partial U} \qquad (1 \text{ variable } C) \tag{5.19}$$

  That is

$$\frac{\partial F_i}{\partial x^j} - \frac{\partial F_j}{\partial x^i} + F_j \frac{\partial F_i}{\partial U} - F_i \frac{\partial F_j}{\partial U} \qquad (1 \text{ variable } C) \tag{5.20}$$

  Alternatively

$$\partial_j F_i - \partial_i F_j + F_j \frac{\partial F_i}{\partial U} - F_i \frac{\partial F_j}{\partial U} \qquad (1 \text{ variable } C) \tag{5.21}$$

  This is one of the most common cases to arise in practice.

- It is sometimes useful to rewrite condition $(C)$ in the equivalent form

$$\frac{\partial F^A{}_i}{\partial x^j} - \frac{\partial F^A{}_j}{\partial x^i} = \sum_{B=1}^{m} \left\{ F^B{}_i \frac{\partial F^A{}_j}{\partial U^B} - F^B{}_j \frac{\partial F^A{}_i}{\partial U^B} \right\} \qquad (C) \tag{5.22}$$

or

$$\partial_j F^A{}_i - \partial_i F^A{}_j = \sum_{B=1}^{m} \left\{ F^B{}_i \, \frac{\partial F^A{}_j}{\partial U^B} - F^B{}_j \, \frac{\partial F^A{}_i}{\partial U^B} \right\} \qquad (C) \tag{5.23}$$

Doing this should focus your attention on conservative vector fields as a possible way of satisfying the integrability constraints.

- A *sufficient* condition for condition $(C)$ to hold in general is that

$$F^A{}_i(x, U) = \frac{\partial \Phi(x)}{\partial x^i} \, J^A(U); \qquad (C2) \tag{5.24}$$

Try it and see. (I explicitly do *not* claim this condition is *necessary.*)

If this sufficient condition holds, then the Frobenius/Mayer system reduces to

$$\frac{\partial U^A}{\partial x^i} = \frac{\partial \Phi(x)}{\partial x^i} \, J^A(U). \tag{5.25}$$

But now we can solve this by reducing it to an ODE. Note that each of the $U^A$, considered as a function of the $x^i$, can change only in the direction parallel to

$$\partial_i \Phi(x) = \frac{\partial \Phi(x)}{\partial x^i}. \tag{5.26}$$

But this means that for some set of functions $\widetilde{U}^A(\Phi)$ we have

$$U^A(x) = \widetilde{U}^A(\Phi(x)), \tag{5.27}$$

with the PDE reducing to

$$\frac{\mathrm{d}\tilde{U}^A(\Phi)}{\mathrm{d}\Phi} = J^A(\tilde{U}). \tag{5.28}$$

This reduces the Frobenius/ Mayer system [subject to this sufficient condition $(C2)$] to an ODE. In fact it is an autonomous ODE, which we already know to be integrable.

- There is an even more special case, obvious given the above discussion, that I will belabour because of its importance: the *autonomous* Frobenius/ Mayer system.

### 5.1.3 Autonomous Frobenius–Mayer systems

**Definition 9 Autonomous Frobenius/Mayer system:**

*The* autonomous **Frobenius/ Mayer** *system is*

$$\frac{\partial U^A}{\partial x^i} = F^A{}_i(U^1, \dots, U^m) \qquad A = 1, 2, \dots, m, \quad i = 1, 2, \dots, n \quad (AF) \tag{5.29}$$

- Note: The key feature is that there is now no explicit $x$ dependence on the RHS.
- The class of autonomous Frobenius/ Mayer systems can be characterized as a particular sub-class of autonomous first-order quasi-linear PDEs.
- The $m$ dependent variables $\{U^A\}$ again depend on the $n$ independent variables $\{x^i\}$.
- All these equations are again of first order.
- There are again as many PDEs as there are first-order derivatives.
- That is, $nm$ of them.
- The RHS now depends only on the dependent variables, the $U$'s.
- There is no *explicit* $x$ dependence on the RHS.
- The equations are "autonomous" in the sense that the "driving term" does not pay any attention to the independent variables, the $x$'s.
- The "driving term" or "source term" now depends only on the "current state" of the system — the $U$'s.

**Theorem 5 The Autonomous Frobenius Integrability Theorem:**

*Suppose the functions* ${F^A}_i(U^A)$ *are smooth functions of all their variables in a neighbourhood of the origin, for* $A = 1, 2, \ldots, m$.

*Then the autonomous Frobenius system (AF) has a unique solution, satisfying the IC*

$$U^A(0, 0, \ldots, 0) = b^A \qquad (k = 1, 2, \ldots, m) \tag{5.30}$$

*for arbitrary given* $b^A$, *if and only if*

$$\sum_{B=1}^{m} {F^B}_i \; \frac{\partial {F^A}_j}{\partial U^B} = \sum_{B=1}^{m} {F^B}_j \; \frac{\partial {F^A}_i}{\partial U^B} \qquad (AC) \tag{5.31}$$

*for all* $i$, $j$, *and* $A$ *in their respective ranges.*

But now let's take a more careful look at the condition $(AC)$.

- If $n = 1$ [so that we are working in one dimension] condition $(AC)$ is always satisfied. But this is just the autonomous version of our previous discussion. After a change in notation ($x \to t$, $U^A \to x^A$) the 1-d autonomous Frobenius equation becomes

$$\frac{\mathrm{d}x^A}{\mathrm{d}t} = F^A(x^B) \qquad\qquad A = 1, 2, \ldots, m. \tag{5.32}$$

- Suppose in contrast that $m = 1$ so there is only one *dependent* variable $U$, only a single "field". Then condition (AC) reduces to

$$F_i \; \frac{\partial F_j}{\partial U} = F_j \; \frac{\partial F_i}{\partial U} \qquad (1 \text{ variable } AC) \tag{5.33}$$

  Then

$$F_i \; \frac{\partial F_j}{\partial U} - F_j \; \frac{\partial F_i}{\partial U} = F_i^2 \; \frac{\partial (F_j/F_i)}{\partial U} = 0 \tag{5.34}$$

  But this is satisfied iff (if and only if) $F_i/F_j = k_i/k_j$ for some set of constants $k_i$ independent of $U$.

  That implies

$$F_i(U) = k_i \; f(U), \tag{5.35}$$

for some constant vector $k_i$.

But this now lets us write the 1-variable integrable autonomous Frobenius system as

$$\frac{\partial U}{\partial x^i} = k_i \; f(U) \qquad\qquad i = 1, 2, \ldots, m. \tag{5.36}$$

Thus the system (if it satisfies condition (AC) so that it is integrable) can be reduced to an ODE in a single variable, call it $\xi$:

$$U(x) = \tilde{U}\,(k \cdot x)\,; \qquad \frac{\mathrm{d}\tilde{U}(\xi)}{\mathrm{d}\xi} = f(\tilde{U}) \tag{5.37}$$

Note that this is all a special case of condition (C2) above.

- In fact for any $n$ and $m$, a *sufficient* condition for condition (AC) to hold is that

$$F^A{}_i(x, U) = k_i \; J^A(U); \qquad (AC2) \tag{5.38}$$

Try it and see. (I do not claim this condition is *necessary*.)

If this sufficient condition holds then the autonomous Frobenius–Mayer system reduces to

$$\frac{\partial U^A}{\partial x^i} = k_i \; J^A(U). \tag{5.39}$$

But we can again solve this by reducing it to an ODE. Note that each of the $U^A$, considered as a function of the $x^i$, can change only in the direction parallel to $k_i$.

But this means that for some set of functions $\widetilde{U}^A(\xi)$ we have

$$U^A(x) = \widetilde{U}^A(\xi); \qquad \xi = \xi_0 + \sum_{i=1}^{m} k_i \; x^i \tag{5.40}$$

with the PDE reducing to

$$\frac{\mathrm{d}\tilde{U}^A(\xi)}{\mathrm{d}\xi} = J^A(\tilde{U}). \tag{5.41}$$

This again reduces the autonomous Frobenius–Mayer system [subject to this sufficient condition (AC2)] to an ODE.

- Clearly the most interesting cases are $n > 1$ and $m > 1$.
- You can have some fun exploring necessary and sufficient conditions, and digging deep into the bowels of the library.

### 5.1.4 Exercises

**Conservative vector fields**

A vector field $V$ is called conservative if curl $V = 0$.
It is a well known fact that if $V$ is conservative on an open subset $W$ of $\mathbb{R}^3$, then there is a function $U(x, y, z)$ such that

$$\vec{V} = -\text{grad } U \tag{5.42}$$

on $W$.
We now want to relate this to the concept of a Frobenius–Mayer system.

a. Show that the system of PDEs that result from

$$\text{grad } U = -\vec{V} \tag{5.43}$$

is a Frobenius system, (a particularly simple Frobenius system), and show that this system can be made to satisfiy the conditions of the Frobenius Complete Integrability theorem.
(What is the consistency condition?)

b. Find the function $U$ if:

i. $\vec{V} = xyz\,\vec{i} + (x^2z/2 - z\sin(yz))\,\vec{j} + (x^2y/2 - y\sin(yz))\,\vec{k}$.

ii. $\vec{V} = (A/r^3)\,\vec{r}$, where A is a constant, $\vec{r} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ is the usual radius vector, $\vec{r}$, and $r = |r|$.

**Height-slope relations**

(A slightly more complicated example; essentially two-dimensional)

Consider now a specific Frobenius theorem with $m = 1$ (so there is only one dependent variable, which I will call $h$) and $n = 2$ (so there are two independent variables, two dimensions, which I shall call $x$ and $y$). Then the Frobenius system is

$$\frac{\partial h(x,y)}{\partial x} = F_x(x,y,h); \qquad \frac{\partial h(x,y)}{\partial y} = F_y(x,y,h). \tag{5.44}$$

You can interpret this, for instance, as the equation for the height of a hill as a function of $x$ and $y$, given that there is a PDE controlling the height of the hill that makes the slope of the hill depend on its height (a self-referential height-slope function).

a. Explicitly write out the set of consistency conditions required for this Frobenius system to have a solution. Ignoring trivial re-labellings, how many non-trivial consistency conditions are there?

b. Now consider the three-dimensional vector

$$\vec{v}(x,y,z) = \Big(F_x(x,y,z),\ F_y(x,y,z),\ 1\Big), \tag{5.45}$$

where now I have relabelled $h \to z$.
Calculate the "vorticity":

$$\vec{\omega} = \text{curl } \vec{v} = \nabla \times \vec{v}. \tag{5.46}$$

Calculate the "helicity":

$$H = \vec{v} \cdot (\text{curl } \vec{v}) = \vec{v} \cdot (\nabla \times \vec{v}). \tag{5.47}$$

c. Show that the condition that the helicity vanishes,

$$H = \vec{v} \cdot (\text{curl } \vec{v}) = 0, \tag{5.48}$$

is equivalent to the Frobenius consistency condition in part [a].
This implies that if the helicity $H$ of $\vec{v}(x,y,z)$ is zero then it is possible to self-consistently find a height function $z(x,y)$ with

$$\partial_i z(x,y) = v_i(x,y,z). \tag{5.49}$$

(This result as given is special to $m = 1$, $n = 2$; there is a generalization of this result to $m = 1$, $n \geq 3$ which is a little tricker to formulate nicely.)

**Autonomous example**

(Fully three-dimensional) Consider the system of PDEs

$$\partial_x U = h_x(U(x,y,z)); \tag{5.50}$$

$$\partial_y U = h_y(U(x,y,z)); \tag{5.51}$$

$$\partial_z U = h_z(U(x,y,z)). \tag{5.52}$$

1. Write down all the Frobenius integrability conditions for this system. How many of the constraints are nontrivial?

2. By adopting the notation

$$\vec{H} = (h_x, h_y, h_z), \tag{5.53}$$

show that the integrability conditions are equivalent to

$$\vec{H} \times \frac{d\vec{H}}{dU} = 0. \tag{5.54}$$

3. Hence show that this system satisfies the integrability conditions iff

$$\vec{H} = \vec{k}\, f(U). \tag{5.55}$$

where $\vec{k}$ is a constant vector.

4. Show that in this situation the solution of the Frobenius system is given by the implicit equation

$$\int_{U_0}^{U} \frac{\mathrm{d}\bar{U}}{f(\bar{U})} = \vec{k} \cdot \vec{x}. \tag{5.56}$$

That is, there exists an invertible function $g(U)$ such that

$$g(U) = \vec{k} \cdot \vec{x}, \tag{5.57}$$

and so

$$U(x) = g^{-1}\left(\vec{k} \cdot \vec{x}\right). \tag{5.58}$$

Indeed

$$\frac{\mathrm{d}g}{\mathrm{d}U} = \frac{1}{f(U)}. \tag{5.59}$$

## Challenges

- **Challenge:** Look up, read, and understand, the proof of the Frobenius–Meyer integrability theorem.

- **Challenge:** Look up, read, and understand, the connection between the Frobenius–Meyer integrability theorem for PDEs and the "Frobenius theorem" of differential geometry.

## 5.2 Two other first-order PDEs

We have already seen several examples of reasonably general classes of first-order PDEs:

- First-order quasi-linear PDEs:

$$\alpha(x, y, U)\, \partial_x U + \beta(x, y, U)\, \partial_y U + \gamma(x, y, U) = 0. \tag{5.60}$$

- The PDE of Cauchy's theorem:

$$\frac{\partial U}{\partial x} = f\left(x, y, U, \frac{\partial U}{\partial y}\right). \tag{5.61}$$

- Frobenius–Mayer systems:

$$\partial_i U^A = F_i^A(x^j, U^B). \tag{5.62}$$

Two other first-order PDEs of considerable importance are briefly discussed below.

### 5.2.1 The continuity equation

The continuity equation is

$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \tag{5.63}$$

- Used wherever there is a "conservation law" for either mass/ charge/ probability.
- Fluid dynamics.
- Electromagnetism.
- Probabilistic modelling; stochastic equations.
  (Physics, Statistics, Finance, Biology, Chemistry, Geology.)

In the general situation you would want to think of the velocity $\vec{v}$ as three additional dependent variables, so that you have

$$\partial_t \rho + (\vec{v} \cdot \vec{\nabla})\rho + \rho\, (\vec{\nabla} \cdot \vec{v}) = 0. \tag{5.64}$$

This is a first-order quasi-linear PDE connecting (in three space dimensions) four dependent variables.

### 5.2.2 The fluid dynamics Euler equation

The hydrodynamic Euler equation is

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho} \tag{5.65}$$

This is actually Newton's second law $\vec{F} = m\ \vec{a}$, but rewritten in terms of individual little blobs of fluid. Here $p$ is the pressure, $\vec{B}$ is any external force (for example, gravity).

For a velocity field $\vec{v}(t, \vec{x})$ the velocity of an individual particle at point $\vec{x}$ at time $t$ is:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \equiv \vec{v}(t, \vec{x}) \tag{5.66}$$

But now take a look at the acceleration:

$$\vec{a} = \frac{\mathrm{d}^2\vec{x}}{\mathrm{d}t^2} = \frac{\mathrm{d}\vec{v}(t, \vec{x}(t))}{\mathrm{d}t} \tag{5.67}$$

But by the chain rule

$$\vec{a} = \partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{5.68}$$

Note the *nonlinearity* in the velocity field.

The hydrodynamic Euler equation is another example of a first-order quasi-linear PDE connecting many dependent variables.

**Exercise 1** *Suppose the pressure is identically zero, a situation generally referred to as "dust", and there are no body forces. Then Euler's hydrodynamic equation reduces to*

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \vec{0}. \tag{5.69}$$

*By whatever means you can, demonstrate that this PDE has the (implicit) general solution:*

$$\vec{v}(t, x) = \vec{f}\Big(\vec{x} - \vec{v}(t, \vec{x})t\Big). \tag{5.70}$$

*Here $\vec{f}(\vec{x})$ is an arbitrary function $\Re^n \to \Re^n$.*

In the next chapter we will use the phrase "Euler equation" in a very different way. The nomenclature, with Euler's name being attached to two such very distinct equations, is unfortunately standard.

# Chapter 6

# The $2^{nd}$ order Euler PDE

The Euler equation of $2^{nd}$ order PDEs is a specific PDE, (*not* the Euler equation of fluid dynamics), that encompasses a wide variety of phenomena — that's why we are going to spend quite some time discussing both it and its general solutions.

## 6.1 Definition

**Definition 10** *The Euler PDE (of $2^{nd}$ order PDEs) is*

$$a\,U_{xx} + 2h\,U_{xy} + b\,U_{yy} = 0 \tag{6.1}$$

*where $a$, $b$, and $h$ are constants.*

*[They could in general be taken as functions of $x$ and $y$, but not yet!].*

We shall rewrite this equation in a form for which the general solution will be obvious.

**Warning 4**
*This is* not *the Euler equation of fluid mechanics.*
*That is a rather different beastie. See previous chapter.*

**Comment 3** *Note that this version of the Euler equation is a linear second-order PDE with constant coefficients.*

## 6.2 Transformation of coordinates

Consider a linear transformation of the coordinates (that is, the independent variables $x$ and $y$) to new independent variables $s$, $t$, defined as follows:

$$s = x + cy; \tag{6.2}$$

$$t = x + dy. \tag{6.3}$$

Note the Jacobian determinant is

$$\det\left(\frac{\partial(s,t)}{\partial(x,y)}\right) = \det\begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} = d - c. \tag{6.4}$$

We shall now rewrite the Euler equation in terms of these new independent variables, and then cunningly choose the parameters $c$ and $d$ so that the resulting equation is really easy to solve.

We have (by the 2-variable chain rule):

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial s}\,\frac{\partial s}{\partial x} + \frac{\partial U}{\partial t}\,\frac{\partial t}{\partial x} = \frac{\partial U}{\partial s} + \frac{\partial U}{\partial t}, \tag{6.5}$$

or equivalently

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}. \tag{6.6}$$

Similarly,

$$\frac{\partial}{\partial y} = c\frac{\partial}{\partial s} + d\frac{\partial}{\partial t}. \tag{6.7}$$

Hence

$$\begin{aligned} U_{xx} &= \left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial x}\right]U && (6.8)\\ &= \left[\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right]U && (6.9)\\ &= U_{ss} + 2U_{st} + U_{tt}. && (6.10) \end{aligned}$$

Similarly

$$U_{yy} = c^2\, U_{ss} + 2cd\, U_{st} + d^2\, U_{tt}; \tag{6.11}$$

and

$$U_{xy} = c\, U_{ss} + (c + d)\, U_{st} + d\, U_{tt}. \tag{6.12}$$

Combining these results we easily see:

$$a\,U_{xx} + 2h\,U_{xy} + b\,U_{yy} = \\ (a + 2hc + bc^2)\,U_{ss} + 2(a + h(c+d) + bcd)\,U_{st} + (a + 2hd + bd^2)\,U_{tt}. \tag{6.13}$$

Leading to the transformed Euler equation (TEE):

$$(a + 2hc + bc^2)\,U_{ss} + 2(a + h(c+d) + bcd)\,U_{st} + (a + 2hd + bd^2)\,U_{tt} = 0. \tag{6.14}$$

## 6.3 Choosing the parameters

# Some very cunning choices (VCC)

To solve the TEE we will make some crafty choices for the parameters $c$ and $d$ occurring in the change of variables. The choices we shall make will depend on the solutions to the quadratic equation

$$a + 2hz + bz^2 = 0. \tag{6.15}$$

We start by supposing that $b$ is nonzero, so this quadratic always has two solutions.

### 6.3.1 Distinct roots

If this equation has two distinct solutions, say $z_1$, $z_2$, then choose the constants $c$ and $d$ to be these solutions:

$$c = z_1; \qquad d = z_2. \tag{6.16}$$

Then we plainly have:

- $c + d =$ the sum of the solutions $= -2h/b$.
- $cd =$ the product of solutions $= ab$.
- The discriminant $4(h^2 - ab) \neq 0$.

Note that since the roots are distinct, the transformation is *proper* (*i.e.*, both the original $x$ and $y$, and the new $s$ and $t$, are independent variables).[1] The Euler equation becomes

$$2\left[a + h\left(-\frac{2h}{b}\right) + b\,\frac{a}{b}\right] U_{st} = 0 \tag{6.17}$$

or

$$2\,\frac{2ab - 2h^2}{b}\,U_{st} = 0. \tag{6.18}$$

That is

$$-\frac{4(h^2 - ab)}{b}\,U_{st} = 0. \tag{6.19}$$

whence, since by hypothesis $h^2 - ab$ is not zero, and $b$ is not zero, we have

$$\boxed{U_{st} = 0.}$$

This transformed PDE is, of course, easy to solve. Its general solution is

$$\boxed{U(s,t) = F(s) + G(t),}$$

where $F$ and $G$ are arbitrary functions. Therefore, as functions of $x$ and $y$:

$$\boxed{U(x,y) = F(x + cy) + G(x + dy).}$$

### 6.3.2 Coincident roots

In this case, the discriminant $4(h^2 - ab) = 0$, and the quadratic has the single solution $z = -h/b$. So let's choose $d$ to be the single root, $d = -h/b$. The last term in the transformed Euler equation (TEE) then vanishes. Furthermore the coefficient of the second term is:

$$a + h(c + d) + bcd = a + hc - \frac{h^2}{b} - \frac{bh}{b}c = 0. \tag{6.20}$$

Hence the TEE reduces to

$$(a + 2hc + bc^2)\,U_{ss} = 0. \tag{6.21}$$

[1]The transformation is *proper* iff the Jacobian determinant is non-zero: That is $\det\left(\frac{\partial(s,t)}{\partial(x,y)}\right) \neq 0$. This happens iff $c \neq d$.

If we choose $c$ to be *different* from $d$ (which we *must* do to keep the transformation *proper*,[2] and so keep the independent variables $s$ and $t$ truly independent of each other) we have

$$\boxed{U_{ss} = 0,}$$

which has the obvious general solution

$$\boxed{U(s,t) = sF(t) + G(t),}$$

where $F$ and $G$ are arbitrary functions. The choice of the value of $c$ is up to you here — it can be anything *except* $d$, the solution to the quadratic. Therefore, as functions of $x$ and $y$:

$$\boxed{U(x,y) = (x + cy)\ F(x + dy) + G(x + dy); \qquad c \neq d.}$$

### 6.3.3 Degenerate quadratic

When $b = 0$, the work above does not apply, as we no longer have a genuine quadratic in $z$. However, you can easily adapt the theory outlined above for a transformation

$$s = cx + y, \tag{6.22}$$

$$t = dx + y, \tag{6.23}$$

leading to the quadratic

$$az^2 + 2hz + b^2 = 0. \tag{6.24}$$

Then so long as $a$ is nonzero, the results indicated above, with the role of $x$ and $y$ interchanged, apply. If it happens that both $a$ and $b$ are zero, then you have the equation $U_{xy} = 0$ to solve: and this is an easy thing to do. (In fact we have already done it.)

[2]The transformation is *proper*, (that is, *invertible*), iff the Jacobian determinant is non-zero: $\det\left(\frac{\partial(s,t)}{\partial(x,y)}\right) \neq 0$. This happens iff $c \neq d$.

## 6.4 Summary

- If $b$ is nonzero:
  - If $h^2 - ab \neq 0$:
$$U(x, y) = F(x + cy) + G(x + dy), \tag{6.25}$$
    where $c$ and $d$ are the distinct solutions to the quadratic equation
$$a + 2hz + bz^2 = 0. \tag{6.26}$$
  - If $h^2 - ab = 0$:
$$U(x, y) = (x + cy)\ F(x + dy) + G(x + dy), \tag{6.27}$$
    where $d$ is the single solution to
$$a + 2hz + bz^2 = 0, \tag{6.28}$$
    and $c$ is any constant not equal to $d$.
- If $a$ is nonzero:
  - If $h^2 - ab \neq 0$:
$$U(x, y) = F(cx + y) + G(dx + y), \tag{6.29}$$
    where $c$ and $d$ are the distinct solutions to the quadratic equation
$$az^2 + 2hz + b = 0. \tag{6.30}$$
  - If $h^2 - ab = 0$:
$$U(x, y) = (cx + y)\ F(dx + y) + G(dx + y), \tag{6.31}$$
    where $d$ is the single solution to
$$az^2 + 2hz + b = 0, \tag{6.32}$$
    and $c$ is any constant not equal to $d$.
- If both $a = 0$ and $b = 0$:
  - The solution is
$$U(x, y) = F(x) + G(y), \tag{6.33}$$
    where $F$ and $G$ are arbitrary.

**Question 1**

*Do we have to do anything special if the roots of the quadratic are complex?*

## 6.5 Euler type

We define the "Euler type" of an Euler PDE by looking at the $2 \times 2$ matrix formed by the coefficients of the second-derivative terms

$$E = \begin{bmatrix} a & h \\ h & b \end{bmatrix}. \tag{6.34}$$

The reason this $2 \times 2$ matrix is interesting is because it can be used to re-write the Euler equation as

$$[\, \partial_x, \, \partial_y \,] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} U = a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} = 0. \tag{6.35}$$

Now consider the determinant of this matrix and use it to classify Euler equations into the three classes:

**Elliptic:** If the determinant $\det(E)$ is positive.

**Parabolic:** If the determinant $\det(E)$ is zero.

**Hyperbolic:** If the determinant $\det(E)$ is negative.

The reason for the terminology will be a bit mysterious at this stage. Note that the determinant $\det(E) = ab - h^2$ is the negative of the discriminant occurring in the quadratic equation we used to simplify the Euler equation when finding the general solution.

Thus for Euler equations we can re-phrase the classification in terms of the algebraic equation:

$$[\, 1, \, z \,] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = a + 2h\, z + b\, z^2 = 0 \tag{6.36}$$

**Elliptic:** If the roots are complex.

**Parabolic:** If the roots are coincident.

**Hyperbolic:** If the roots are real.

Once you go through the analysis leading to the general solution this leads to the characterization:

**Elliptic:** If the general solution involves arbitrary functions of two distinct complex variables.

**Parabolic:** If the general solution involves arbitrary functions of only one real variable.

**Hyperbolic:** If the general solution involves arbitrary functions of two distinct real variables.

**Warning 5** *I should warn you that while these words Elliptic/ Parabolic/ Hyperbolic are most commonly used within the context of Euler's equation, (and its generalization with first order and linear terms as will be discussed below), the notion is much more general.*

*Extending the Elliptic/ Parabolic/ Hyperbolic distinction to variable coefficients (so that the matrix $E(x, y)$ is position dependent) is easy.*

*Extending it to more dimensions is also easy.*

**Warning 6** *It is less straightforward, but sometimes still possible and useful, to extend the Elliptic/ Parabolic/ Hyperbolic distinction to nonlinear PDEs and to systems of PDEs. See, for instance, Courant and Hilbert for details.*

## 6.6 Challenges

For a challenge here's a few questions to think about — be prepared to do some internet searching...

**Question 2 Terminology:**
*What is the origin of the terminology Elliptic/ Parabolic/ Hyperbolic?*

**Question 3 Terminology:**
*Are the terms Elliptic/ Parabolic/ Hyperbolic exclusive?*

**Question 4 Terminology:**
*Are the terms Elliptic/ Parabolic/ Hyperbolic complete?*
*(Do they cover all the possibilities?)*

**Question 5 Eikonal:**
*What is the meaning of the word "eikonal"?*

**Question 6 Symbol:**
*What is the "symbol" of a PDE?*

**Question 7 Fresnel equation:**
*What is the "Fresnel equation" of a PDE?*

## 6.7 Exercises

### 6.7.1 Euler type

Determine the Euler type (*i.e.*, elliptic, hyperbolic, or parabolic) of each of the following PDEs, and obtain the general solution in each case:

a. $3U_{xx} + 4U_{xy} - U_{yy} = 0.$

b. $U_{xx} - 2U_{xy} + U_{yy} = 0.$

c. $4U_{xx} + U_{yy} = 0.$

d. $U_{xx} + 4U_{xy} + 4U_{yy} = 0.$

e. $U_{yy} + 2U_{xx} = 0.$

f. $4U_{xx} + U_{yy} = 0.$

g. $4U_{,xx} - U_{,yy} = 0.$

h. $4U_{,xx} + U_{,xy} + U_{,yy} = 0.$

i. $9U_{,xx} + 3U_{,xy} + U_{,yy} = 0.$

j. $8U_{,xx} + 3U_{,xy} + U_{,yy} = 0.$

### 6.7.2 General solution to Euler's equation

Find the general solution to the partial differential equation

$$4U_{,xx} + 2U_{,xy} + U_{,yy} = 0, \tag{6.37}$$

in terms of two arbitrary functions.

Now repeat this exercise for all of items [a.] to [j.] above.

### 6.7.3 Generalized constant-coefficient Euler PDE

**Definition 11**
*One simple way of generalizing the Euler PDE is this:*

$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} + c\, U_x + d\, U_y + e\, U + f = 0, \tag{6.38}$$

*where $a$, $b$, $h$, and $c$, $d$, $e$, $f$ are constants (and at least one of the second-order coefficients $a$, $b$, or $h$, is nonzero).*

**Comment 4**
*This is still a linear second-order PDE with constant coefficients.*

This generalization is not really as painful as it looks. If the coefficients are constants the general solution can *sometimes* be found using modifications of the preceding argument. Even then, sometimes there is no closed-form general solution, even for this constant coefficient case.

**Project 1 Generalized constant-coefficient Euler PDE:**
*Analyze this generalized constant-coefficient Euler PDE in detail.*

*Completely classify those situations for which closed-form general solutions (in terms of two arbitrary functions) can be written down.*

*Even when completely general solutions cannot be explicitly written down, it is often possible to find reasonably general classes of specific solution.*

*Do as much as possible...*

### 6.7.4 Specific variable-coefficient extension of Euler's equation

Show that

$$u(x,y) = f(2x + y^2) + g(2x - y^2), \tag{6.39}$$

is a general solution to the equation

$$y^2 u_{xx} + \frac{1}{y} u_y - u_{yy} = 0, \tag{6.40}$$

where $f$ and $g$ are arbitrary differentiable functions.

This is a specific example of a variable-coefficient extension of the Euler equation. Is it elliptic, parabolic, or hyperbolic?

*We will have a lot more to say about this class of PDEs later.*

### 6.7.5 Tricomi's equation

Consider Tricomi's PDE:

$$y\, U_{xx} + U_{yy} = 0. \tag{6.41}$$

Is it elliptic, parabolic, or hyperbolic?

Try to find a general solution to this PDE...

(Don't be surprised to find it's impossible, at least at this stage of the course. By the end of the course you will see techniques powerful enough to write down a general solution for this PDE.)

*We will have a lot more to say about this class of PDEs later.*

# Chapter 7

# Euler PDEs: Standard examples

I'll now give a catalogue of standard examples of Euler PDEs that you should learn to recognize.

## 7.1 The Wave Equation

(Typical example of a hyperbolic PDE).

$$\boxed{U_{xx} - \frac{1}{c^2}\, U_{tt} = 0.}$$

- Here, $U(x,t)$ represents the displacement at point $x$ and at time $t$ of a string from its equilibrium position.
- That is, $U(x,t)$ is the shape of the string at time $t$.
- The constant $c$ is the velocity of the wave disturbance.
- The same equation can be used to describe sound waves or light waves; at least in flat spacetime.
- The generalizations to get to curved spacetime are not too onerous, but not appropriate for Math 301.

- Usually you know:
  - That the string is fixed at the origin $x = 0$ and at the end point $A$ ($x = L$, say).
  - The initial shape of the string, $U(x, 0)$.
  - The velocity of each point $x$ of the string, $U_t(x, 0)$.
    [most often this will be zero, the string will start from rest]
  - Conditions of this sort, where you know initial values of the function and its derivatives, are called Cauchy (initial) conditions.
  - It can be shown that Cauchy initial conditions are necessary and sufficient for the existence and uniqueness of solutions.

    (This is typical of those problems that are classified as hyperbolic — Cauchy conditions are enough to guarantee existence and uniqueness of solutions).

- In terms of the Euler PDE

  $$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} = 0, \tag{7.1}$$

  the wave equation corresponds to

  $$a \to 1; \quad h \to 0; \quad b \to -\frac{1}{c^2}, \tag{7.2}$$

  with the notational change $y \to t$.

- Without further calculation we can use the analysis of the Euler PDE to immediately write down the general solution of the wave equation:

  $$U(x, t) = f(x - ct) + g(x + ct). \tag{7.3}$$

  This is d'Alembert's solution, and I'll have considerably more to say about it later.

## 7.2 The Heat or Diffusion equation

(Typical example of a parabolic PDE).

$$\boxed{U_t = \sigma\, U_{xx}.}$$

- Here $\sigma$ is a constant, called the thermal diffusivity (heat equation) or simply the diffusion constant.
- Such an equation often occurs in situations where diffusion occurs. For example, consider a heated bar of metal:

  $U(x,t)$ is the temperature at time $t$ at a point $x$ along the bar.

  You might be given:

  - the initial distribution of temperature in the bar, $U(x,0)$.
  - or, that the two ends of the bar are kept a fixed temperatures,

    $$U(0,t) = T_1, \tag{7.4}$$
    $$U(L,t) = T_2, \tag{7.5}$$

    where $L$ is the length of the bar.

  Then again you might be told:

  - the initial distribution of temperature in the bar, $U(x,0)$.
  - or, the fact that the ends are insulated, so that no heat can pass through them:

    $$U_x(0,t) = 0 = U_x(L,t) \quad \text{for all } t. \tag{7.6}$$

- Typically, for parabolic equations, conditions of the type described above will guarantee the existence and uniqueness of a solution.
- In terms of the generalized Euler PDE

  $$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} + c\, U_x + d\, U_y + e\, U + f = 0 \tag{7.7}$$

  the heat equation corresponds to

  $$a \to \sigma; \quad h \to 0; \quad b \to 0; \tag{7.8}$$

  $$c \to 0; \quad d \to -1; \quad e \to 0; \quad f \to 0 \tag{7.9}$$

  with the notational change $y \to t$.

- There is no closed-form general solution in terms of arbitrary functions, (at least not without doing some extra integrals), but we will later in the course use Fourier series to give a general solution in terms of an infinite series of suitable "basis functions".

- Here is another trick:

$$U(x,t) = \frac{\exp(-\frac{x^2}{4\sigma t})}{\sqrt{t}} \tag{7.10}$$

satisfies the heat equation for $t > 0$.

Check:

$$U_{xx} = \left( -\frac{1}{2\sigma t} + \frac{x^2}{4\sigma^2 t^2} \right) U; \qquad U_t = \left( -\frac{1}{2t} + \frac{x^2}{4\sigma t^2} \right) U. \tag{7.11}$$

So yes,

$$U_t = \sigma \, U_{xx}. \tag{7.12}$$

This is a very specific solution to the heat equation, but we shall later on see how to build up other interesting solutions based on this specific solution.

- Let $x_0$ and $t_0$ be arbitrary constants. Show that

$$U(x,t;x_0,t_0) = \frac{\exp(-\frac{(x-x_0)^2}{4\sigma(t-t_0)})}{\sqrt{t-t_0}} \tag{7.13}$$

satisfies the heat equation for $t > t_0$.

- Hence show that for any sufficiently well behaved function $s(x_0,t_0)$ the function

$$U(x,t) = \int_{-\infty}^{0} \int_{-\infty}^{+\infty} s(x_0,t_0) \, \frac{\exp(-\frac{(x-x_0)^2}{4\sigma(t-t_0)})}{\sqrt{t-t_0}} dx_0 dt_0 \tag{7.14}$$

satisfies the heat equation for $t > 0$.

## 7.3 Laplace's equation

(Typical of an elliptic PDE).

$$\boxed{U_{xx} + U_{yy} = 0.}$$

- Now $U(x, y)$ represents, for example,
  - the electrostatic potential at the point $(x, y)$ in a piece $R$ of dielectric medium,
  - or the Newtonian gravitational potential in empty space (outside the sources),
  - or it might represent the equilibrium temperature at the point $(x, y)$ inside a heated solid $R$.
- Typically, in problems involving Laplace's equation, boundary conditions of the following form are known:
  1. You might be given the potential (temperature) on the boundary $B = \partial R$ of the region $R$:
     $$U(x, y) \quad \text{is given on } B. \tag{7.15}$$
     Such a condition is called a Dirichlet condition.
  2. You might know the flux of $U$ (that is, the gradient of $U$ normal to the boundary $B$) into the region $R$ :
     $$\frac{\partial U}{\partial n} \quad \text{is given on } B. \tag{7.16}$$
     Such a condition is called a Neumann condition.
  3. Frequently, you might be given a mixture of Dirichlet and Neumann conditions. (Robin boundary conditions.)
- So long as the boundary shape $B$ is "reasonable", you can be sure there will be a unique solution to Laplace's equation satisfying any of these boundary conditions.
- In terms of the Euler PDE
  $$a\,U_{xx} + 2h\,U_{xy} + b\,U_{yy} = 0, \tag{7.17}$$
  the Laplace equation corresponds to
  $$a \to 1; \quad h \to 0; \quad b \to 1. \tag{7.18}$$

- Without further calculation we can use the analysis of the Euler PDE to immediately write down the general solution of Laplace's equation:

$$U(x, y) = f(x + iy) + g(x - iy). \tag{7.19}$$

This is Laplace's solution, which relates the solution of the Laplace PDE to the theory of functions of a complex variable. I'll also have more to say about this later.

## 7.4 Review: Elliptic/ Parabolic/ Hyperbolic

### 7.4.1 Euler PDE versus Laplace PDE:

- When is the Euler differential equation elliptic?
- When is the Euler differential equation qualitatively similar to Laplace's equation?
- When is it qualitatively different?

### 7.4.2 Euler PDE versus Wave PDE:

- When is the Euler differential equation hyperbolic?
- When is the Euler differential equation qualitatively similar to the wave equation?
- When is it qualitatively different?

### 7.4.3 d'Alembert's solution

What is the general solution of the wave equation

$$U_{tt} = c^2 U_{xx} \tag{7.20}$$

in terms of two arbitrary functions?

### 7.4.4 Laplace's solution.

What is the general solution of Laplace's equation

$$U_{xx} + U_{yy} = 0 \tag{7.21}$$

in terms of two arbitrary functions?

## 7.5 Other standard Euler PDEs

Additional examples of PDEs of the generalized constant-coefficient Euler class are:

- Klein–Gordon equation:

$$\partial_t^2 \phi - \nabla^2 \phi = -m^2 \phi. \tag{7.22}$$

  - This generalizes the wave equation.
  - In particle physics, suitable for a scalar particle with mass.

    (For example, the Higgs particle after spontaneous symmetry breaking in the standard model of particle physics.)
  - In plasma physics, where it is useful for describing screening effects. ($m \longleftrightarrow$ Debye screening length.)
  - Also used in super-conductivity — $m$ is then related to the London flux penetration depth.
  - Useful for a string in a valley.
  - A specific solution:

$$\phi(x,t) = \sin\left(\sqrt{m^2+k^2}\; t + kx + \varphi\right) \tag{7.23}$$

  In terms of the generalized Euler PDE

$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} + c\, U_x + d\, U_y + e\, U + f = 0, \tag{7.24}$$

  the Klein–Gordon equation corresponds to

$$a \to 1; \quad h \to 0; \quad b \to -1; \tag{7.25}$$

$$c \to 0; \quad d \to 0; \quad e \to m^2; \quad f \to 0, \tag{7.26}$$

  with the notational change $x \to t$, $y \to x$.

  There is also a natural generalization from (1+1) to (2+1) and (3+1) dimensions.

- Helmholtz equation:

$$\nabla^2 \phi = m^2 \phi. \tag{7.27}$$

  - Generalizes Laplace's equation.

– Often results from the wave equation after "separation of variables" — lots more on this later!
– Also used in early nuclear physics — the pion potential.
– In 3 dimensions a specific solution is:

$$\phi = \frac{\exp(-mr)}{r}. \tag{7.28}$$

$$\nabla\phi = -(1+mr)\exp(-mr)\,\frac{\hat{r}}{r^2}. \tag{7.29}$$

– Note exponential modification of "inverse square" law.

In terms of the generalized Euler PDE

$$a\,U_{xx} + 2h\,U_{xy} + b\,U_{yy} + c\,U_x + d\,U_y + e\,U + f = 0, \tag{7.30}$$

the Helmholtz equation corresponds to

$$a \to 1; \quad h \to 0; \quad b \to 1; \tag{7.31}$$

$$c \to 0; \quad d \to 0; \quad e \to m^2; \quad f \to 0. \tag{7.32}$$

There is a natural generalization to three space dimensions.

- Maxwell equations (source free):

$$\text{div } E = 0; \tag{7.33}$$
$$\text{curl } B - \partial_t E = 0; \tag{7.34}$$
$$\text{div } B = 0; \tag{7.35}$$
$$\text{curl } E + \partial_t B = 0. \tag{7.36}$$

These four PDEs link the space and time dependence of electric and magnetic fields. (Thankfully they are linear PDEs, which is why we can do such a lot with them.) These equations are very well understood and underly much of humanity's pre-quantum technology.

The Maxwell equations can be put into the form of a *system* of Euler PDEs, with electric fields coupled to magnetic fields.

For a small challenge, use the rules of vector calculus to derive wave equations for $E$ and $B$:

$$\partial_t^2 E - \nabla^2 E = 0; \tag{7.37}$$

$$\partial_t^2 B - \nabla^2 B = 0. \tag{7.38}$$

Note that for simplicity I have adopted units where the speed of light equals unity.

For a small challenge, write $F = E + iB$ and show that the (source free) Maxwell equations reduce to

$$\text{div } F = 0; \qquad \text{curl } F = i\,\dot{F} \tag{7.39}$$

By now I hope you are convinced of the central importance of the Euler PDE, both in its original form and in the generalized constant-coefficient case. (And later on we'll see even more generalizations.)

# Chapter 8

# The d'Alembert solution

## 8.1 General solution and boundary conditions

Suppose $U(x,t)$ satisfies the wave equation

$$\boxed{U_{xx} - \frac{1}{c^2}\, U_{tt} = 0,}$$

and suppose that the BC (actually, IC) are:

$$U(x,0) = f(x), \tag{8.1}$$

$$U_t(x,0) = g(x). \tag{8.2}$$

For example, $U(x,t)$ could be the displacement of an infinitely long stretched string set vibrating from its equilibrium position along the $X$-axis by starting it off with the shape defined by $f(x)$ and the velocity $g(x)$.
The general solution to this equation is

$$U(x,t) = F(x+ct) + G(x-ct), \tag{8.3}$$

where $F$ and $G$ are arbitrary functions.

- But we do not at this stage know what $F$ and $G$ look like in terms of our "given" data, $f$ and $g$, that is the problem we will now solve.

Applying the initial conditions, we have

$$F(x) + G(x) = f(x), \tag{8.4}$$

$$c\,F'(x) - c\,G'(x) = g(x), \tag{8.5}$$

where $'$ denotes derivative.

These equations can be solved to find

$$F(x) = \frac{1}{2c}\int_a^x g(s)\,\mathrm{d}s + \frac{1}{2}f(x), \qquad (a \text{ is arbitrary}) \tag{8.6}$$

and

$$G(x) = -\frac{1}{2c}\int_a^x g(s)\,\mathrm{d}s + \frac{1}{2}f(x), \tag{8.7}$$

so that the general solution, when constrained in terms of the two initial conditions $f$ and $g$, is:

$$\boxed{U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.}$$

If, for example, the string was released from rest, so that $g(x) = 0$, and if its initial displacement is $f(x)$, we find

$$\boxed{U(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$$

which shows that the displacement travels down the string both ways, keeping its shape, with velocity $c$. Thus we can interpret the constant $c$ in the wave equation as the velocity of the ensuing waves of vibration.

**Question 8**
*How does this generalize to more than one space dimension?*
*In fact,* does *this generalize to more than one space dimension?*

## 8.2 Difficulties with d'Alembert's solution

Many problems have equations for which the general solution is easy to find, but for which other conditions (boundary conditions, initial condition) make it impossible to solve by using this general solution. In d'Alembert's solution, for example, we knew the initial shape of the string, so we required our general solution to also satisfy the Boundary Conditions, leading to the functional equations:

$$F(x) + G(x) = f(x), \tag{8.8}$$

$$c\,F'(x) - c\,G'(x) = g(x), \tag{8.9}$$

for the arbitrary functions $F$ and $G$.

In d'Alembert's case, these were easy equations to solve: but in many other cases, the functional equations that result from the BC are extremely difficult to solve. (This is one of the reasons why we are not too concerned if it's not possible to write down a "general solution" for a given PDE. Though nice to have available, once you apply boundary conditions "general solutions" are not always as useful as one might hope.)

Consult the second heat equation example of the of the SOV problems/ exercises below. There, the general solution is obvious, but the equations resulting from the boundary conditions are practically impossible to solve.

## 8.3 Exercises

### 8.3.1 An extension to d'Alembert's solution

Show that

$$U(x, y, t) = F(x + iay - vt) + G(x - iay - vt), \tag{8.10}$$

where $F$ and $G$ are arbitrary twice differentiable functions, is a general solution (in the sense we have defined it) of the (2+1) dimensional wave equation

$$U_{xx} + U_{yy} = \frac{1}{c^2} U_{tt}, \tag{8.11}$$

when

$$a^2 = 1 - v^2/c^2. \tag{8.12}$$

Show that this can be rewritten ($\theta$ is a constant) as

$$U(x, y, t) = \tilde{F}(x\cos\theta + iy\sin\theta - ct) + \tilde{G}(x\cos\theta - iy\sin\theta - ct), \tag{8.13}$$

### 8.3.2 Applying d'Alembert's solution

Solve the wave equation $U_{xx} - U_{tt} = 0$ given that $U(x, 0) = B(x)$ and $U_t(x, 0) = 0$, where $B(x)$ is the bump function

$$B(x) = \begin{cases} 1 & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases} \tag{8.14}$$

Sketch the shape of $U(x, t)$ at some future times $t > 0$; say $t = 2$, 4, 6, and 8. What is the wave velocity?

# Chapter 9

# Euler equation with variable coefficients

## 9.1 Definitions

It is often useful to consider a further extension of the definition of the Euler PDE:

**Definition 12**
*The generalized variable-coefficient Euler PDE is*

$$a(x,y)\,U_{xx} + 2h(x,y)\,U_{xy} + b(x,y)\,U_{yy} + c(x,y)\,U_x + d(x,y)\,U_y$$
$$+e(x,y)\,U + f(x,y) = 0, \tag{9.1}$$

*where $a$, $b$, $h$, and $c$, $d$, $e$, $f$ are now* functions *of $x$ and $y$.*

*(And at least one of the second-order coefficients $a$, $b$, or $h$, is not identically zero.)*

- This is not really as painful as it looks.
- Note that this is simply another name for the most general linear second-order PDE.

- First let's simultaneously focus attention on the second-order derivatives, and generalize the Euler equation even further by allowing for a nonlinear source term. Consider the form below.

**Definition 13**
*The generalized variable-coefficient Euler PDE (with non-linear source) is*

$$a(x,y)\,U_{xx} + 2h(x,y)\,U_{xy} + b(x,y)\,U_{yy} = F(x,y,U,U_x,U_y), \tag{9.2}$$

*where $a$, $b$, and $h$ are functions of $x$ and $y$, and $F$ is a function of its indicated arguments.*

*(And at least one of the second-order coefficients $a$, $b$, or $h$, is not identically zero.)*

- This is still *less general* than the class of quasi-linear Euler PDEs, see below.

## 9.2 Canonical form

A remarkable result in 2-dimensions is that by a change of coordinates the variable coefficients of the second-order terms can *always* be made constant, and the Euler equation can *always* be brought into a simple canonical form.

**Theorem 6**
*In 2 dimensions, as long as $a(x,y)$, $h(x,y)$, and $b(x,y)$ are not all zero, you can always divide the plane into disjoint regions in each of which you can, by change of independent variables, bring the generalized variable-coefficient Euler PDE*

$$a(x,y)\, U_{xx} + 2h(x,y)\, U_{xy} + b(x,y)\, U_{yy} = F(x,y,U,U_x,U_y), \tag{9.3}$$

*into the form*

$$U_{xx} + \epsilon\, U_{yy} = \tilde{F}(x,y,U,U_x,U_y), \tag{9.4}$$

*where $\epsilon = \pm 1$ or $0$, and $\tilde{F}$ is a function of its indicated arguments.*

*Furthermore*

$$\epsilon = sign\left[a(x,y)\, b(x,y) - h(x,y)^2\right]. \tag{9.5}$$

- This theorem generalizes what we are able to do with the constant-coefficient case.
- The existence of this theorem is one of the reasons the 2-dimensional Laplace and wave equations are of such fundamental importance.
- Note that

$$\det \begin{bmatrix} a(x,y) & h(x,y) \\ h(x,y) & b(x,y) \end{bmatrix} = a(x,y)\, b(x,y) - h(x,y)^2 \tag{9.6}$$

  can still be used to classify the PDE as elliptic, parabolic, or hyperbolic, but that this is now a position-dependent classification — the Euler type of the PDE can *change* from one part of the plane to another.

**Proof of the canonical form theorem:**
Consider a change of variables from $x$, $y$ to $\bar{x}$, $\bar{y}$. Let

$$\bar{x} = \phi(x,y); \qquad \bar{y} = \psi(x,y). \tag{9.7}$$

Assume the change of variables is invertible (at least locally) so that

$$x = \Phi(\bar{x},\bar{y}); \qquad y = \Psi(\bar{x},\bar{y}). \tag{9.8}$$

By the inverse function theorem this will be true as long as the Jacobian is nonzero. That is, as long as

$$\frac{\partial(\bar{x},\bar{y})}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = \phi_x\psi_y - \phi_y\psi_x \neq 0. \tag{9.9}$$

Then

$$U(x,y) = U(\Phi(\bar{x},\bar{y}),\Psi(\bar{x},\bar{y})) = \bar{U}(\bar{x},\bar{y}). \tag{9.10}$$

Applying the chain rule:

$$U_x = \bar{U}_{\bar{x}}\ \phi_x + \bar{U}_{\bar{y}}\ \psi_x; \tag{9.11}$$

$$U_y = \bar{U}_{\bar{y}}\ \phi_y + \bar{U}_{\bar{y}}\ \psi_y. \tag{9.12}$$

Differentiating a second time:

$$U_{xx} = \bar{U}_{\bar{x}\bar{x}}\ \phi_x^2 + 2\bar{U}_{\bar{x}\bar{y}}\ \phi_x\psi_x + \bar{U}_{\bar{y}\bar{y}}\ \psi_x^2 + \bar{U}_{\bar{x}}\ \phi_{xx} + \bar{U}_{\bar{y}}\ \psi_{xx}; \tag{9.13}$$

$$U_{xy} = \bar{U}_{\bar{x}\bar{x}}\ \phi_x\phi_y + \bar{U}_{\bar{x}\bar{y}}\ (\phi_x\ \psi_y + \psi_x\ \phi_y) + \bar{U}_{\bar{y}\bar{y}}\ \psi_x\psi_y + \bar{U}_{\bar{x}}\ \phi_{xy} + \bar{U}_{\bar{y}}\ \psi_{xy}; \tag{9.14}$$

$$U_{yy} = \bar{U}_{\bar{x}\bar{x}}\ \phi_y^2 + 2\bar{U}_{\bar{x}\bar{y}}\ \phi_y\psi_y + \bar{U}_{\bar{y}\bar{y}}\ \psi_y^2 + \bar{U}_{\bar{x}}\ \phi_{yy} + \bar{U}_{\bar{y}}\ \psi_{yy}. \tag{9.15}$$

Now add and collect terms to obtain

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = \bar{a}\ \bar{U}_{\bar{x}\bar{x}} + 2\bar{h}\ \bar{U}_{\bar{x}\bar{y}} + \bar{b}\ \bar{U}_{\bar{y}\bar{y}} + \bar{e}\ \bar{U}_{\bar{x}} + \bar{f}\ \bar{U}_{\bar{y}}, \tag{9.16}$$

where

$$\bar{a} = a\ \phi_x^2 + 2h\ \phi_x\phi_y + b\ \phi_y^2; \tag{9.17}$$

$$\bar{h} = a\ \phi_x\ \psi_x + 2h\ (\phi_x\phi_y + \psi_x\phi_y) + b\ \phi_y\psi_y; \tag{9.18}$$

$$\bar{b} = a\ \psi_x^2 + 2h\ \psi_x\psi_y + b\ \psi_y^2; \tag{9.19}$$

$$\bar{e} = a\ \phi_{xx} + 2h\ \phi_{xy} + c\ \phi_{yy}; \tag{9.20}$$

$$\bar{f} = a\ \psi_{xx} + 2h\ \psi_{xy} + c\ \psi_{yy}. \tag{9.21}$$

This turns the original PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = F(x,y,U,U_x,U_y), \tag{9.22}$$

into the form

$$\bar{a}\ \bar{U}_{\bar{x}\bar{x}} + 2\bar{h}\ \bar{U}_{\bar{x}\bar{y}} + \bar{b}\ \bar{U}_{\bar{y}\bar{y}} = F_2(\bar{x},\bar{y},\bar{U},\bar{U}_{\bar{x}},\bar{U}_{\bar{y}}), \tag{9.23}$$

but *we now have the freedom to* choose $\phi$ *and* $\psi$ *to make the transformed coefficients* $\bar{a}$, $\bar{h}$, *and* $\bar{c}$, *as simple as possible.*

Start by choosing $\phi_x$ and $\phi_y$ so that $a\ \phi_x + h\ \phi_y \neq 0$ and $\bar{a} \neq 0$;
this can always be done.
Then choose $\psi_y \neq 0$, and solve for $\bar{h} = 0$. This requires

$$\psi_x = -\psi_y\ \frac{h\phi_x + b\psi_y}{a\phi_x + h\phi_y}. \tag{9.24}$$

We can check that these choices make sense by computing the Jacobian

$$\frac{\partial(\bar{x}, \bar{y})}{\partial(x, y)} = \phi_x\psi_y - \phi_y\psi_x = \frac{\psi_y}{a\phi_x + h\phi_y}\ (a\ \phi_x^2 + 2h\ \phi_x\phi_y + b\ \phi_y^2) = \frac{\psi_y}{a\phi_x + h\phi_y}\ \bar{a}, \tag{9.25}$$

which is nonzero by hypothesis. But then $\bar{b}$ is easily computed to be

$$\bar{b} = \frac{\psi_y^2}{(a\phi_x + h\phi_y)^2}\ (ab - h^2)\ \bar{a}. \tag{9.26}$$

So at this stage we have $\bar{h} = 0$ and

$$\mathrm{sign}(\bar{b}) = \mathrm{sign}(ab - h^2)\ \mathrm{sign}(\bar{a}). \tag{9.27}$$

But the only thing we have used about $\psi_y$ is that it is nonzero, so we are still free to pick

$$\psi_y = \frac{a\phi_x + h\phi_y}{\sqrt{|ab - h^2|}}. \tag{9.28}$$

But then we have both $\bar{h} = 0$ and

$$\bar{b} = \mathrm{sign}(ab - h^2)\ \bar{a}. \tag{9.29}$$

So in this particular coordinate system the PDE is

$$\bar{a}(\bar{x}, \bar{y}) \left\{ \bar{U}_{\bar{x}\bar{x}} + \mathrm{sign}(ab - h^2)\ \bar{U}_{\bar{y}\bar{y}} \right\} = F_2(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}). \tag{9.30}$$

Dividing through by $\bar{a}$ now yields

$$\bar{U}_{\bar{x}\bar{x}} + \mathrm{sign}(ab - h^2)\ \bar{U}_{\bar{y}\bar{y}} = F_3(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}). \tag{9.31}$$

Now adopt the notation

$$\epsilon = \mathrm{sign}(ab - h^2), \tag{9.32}$$

and drop the over-bars to obtain

$$U_{xx} + \epsilon\, U_{yy} = \tilde{F}(x, y, U, U_x, U_y), \tag{9.33}$$

and we are done. QED!

- Note that this works in *any* two-dimensional region where $(ab - h^2)$ is of *constant sign*.

- This includes two dimensional regions where $(ab - h^2)$ is identically zero.

- Note that this is a relatively "straightforward" extension of what we did for the constant-coefficient Euler equation.

- If you want to consider a two dimensional region where $(ab - h^2)$ *changes sign*, then the trick is to use $(ab - h^2)$ as one of your new coordinates, say $\bar{x}$. You can still eliminate $\bar{h}$ in the same way, but now

  $$\bar{b} = \frac{\psi_y^2}{(a\phi_x + h\phi_y)^2}\,(ab - h^2) \times \bar{a} \quad \to \quad \frac{\psi_y^2}{(a\phi_x + h\phi_y)^2}\,\bar{x}\,\bar{a}, \tag{9.34}$$

  and the further choice

  $$\psi_y = a\phi_x + h\phi_y, \tag{9.35}$$

  now leads to

  $$\bar{a}(\bar{x}, \bar{y})\left\{\bar{U}_{\bar{x}\bar{x}} + \bar{x}\,\bar{U}_{\bar{y}\bar{y}}\right\} = F_2(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}). \tag{9.36}$$

  We now rewrite this as

  $$U_{xx} + x\, U_{yy} = \tilde{F}(x, y, U, U_x, U_y), \tag{9.37}$$

  which is Tricomi's equation with a nonlinear source term.

- Note what we have done — in two dimensions the second-derivative part of the general variable-coefficient Euler equation has been reduced to a very small number of standard cases — the wave equation (with nonlinear source), Laplace's equation (with nonlinear source), a parabolic equation (with nonlinear source), or Tricomi's equation (with nonlinear source). This is a tremendous simplification.

- This whole discussion can be given a "geometrical" interpretation which will not make any sense until know some differential geometry (Math 464):
  - Any two-dimensional manifold with a non-singular metric tensor is locally conformally flat.
  - Any two-dimensional manifold with a Euclidean metric tensor is locally conformal to two-dimensional Euclidean space.
  - Any two-dimensional manifold with a Lorentzian metric tensor is locally conformal to two-dimensional Minkowski space.
- Unfortunately if you go beyond 2 dimensions things get a whole lot more complicated.
  - In 3 dimensions you can at least diagonalize the matrix of coefficients of the second-order terms, but you *cannot* make the coefficients piecewise constant.

    (Proving this is *not* easy.)
  - In 4 or more dimensions you cannot even always diagonalize the matrix of coefficients of the second-order terms.

    (Proving this is relatively easy but way outside the scope of this course.)
  - The elliptic/ parabolic/ hyperbolic distinction can still be made but now requires more information than just the determinant of the matrix of second order coefficients — you now need to know the *signature* of that matrix, the number of positive, negative, and zero eigenvalues.
    - If all the eigenvalues of the matrix of second-order coefficients are nonzero and have the same sign, then the PDE is elliptic.
    - If all the eigenvalues of the matrix of second-order coefficients are nonzero and some have differing sign, then the PDE is hyperbolic.
    - If all the eigenvalues of the matrix of second-order coefficients are nonzero and exactly one has a different sign from all the others, then the PDE is strictly hyperbolic.

* If all the eigenvalues of the matrix of second-order coefficients are nonzero and at least two are positive while at least two are negative (which can only happen in four or more dimensions), then the PDE is ultra-hyperbolic.
* If some of the eigenvalues of the matrix of second-order coefficients are zero, then the PDE is elliptic.

## 9.3 Examples

Here are some examples of standard PDEs of considerable importance that fall under the heading of variable-coefficient Euler type.

- Poisson equation:
$$\nabla^2 \phi = \rho. \tag{9.38}$$
Laplace's equation with a position-dependent source.
  - Electrostatic potential in the presence of electric charge.
  - Gravitational potential in the presence of matter.
  - Equilibrium temperature in the presence of heat sources.

  In terms of the generalized Euler PDE
$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} + c\, U_x + d\, U_y + e\, U + f = 0, \tag{9.39}$$
the Poisson equation corresponds to
$$a \to \sigma; \quad h \to 0; \quad b \to 1; \tag{9.40}$$
$$c \to 0; \quad d \to 0; \quad e \to 0; \quad f \to \rho(x, y). \tag{9.41}$$
There is a natural generalization to three space dimensions.
- Maxwell equation (with sources):
Adding charges and currents to the Maxwell equations
$$\text{div } E = \rho; \tag{9.42}$$
$$\text{curl } B - \partial_t E = j; \tag{9.43}$$
$$\text{div } B = 0; \tag{9.44}$$
$$\text{curl } E + \partial_t B = 0. \tag{9.45}$$

In the presence of sources the Maxwell equations can be put into the form of a *system* of generalized variable-coefficient Euler PDEs, with electric fields coupled to magnetic fields, charges, and currents. You can use the rules of vector calculus to derive wave equations for $E$ and $B$:

$$\partial_t^2 E - \nabla^2 E = \text{grad } \rho - \partial_t j; \tag{9.46}$$

$$\partial_t^2 B - \nabla^2 B = -\text{curl } j. \tag{9.47}$$

You might find this form more intuitive:

$$(-\partial_t^2 + \nabla^2) E = -\text{grad } \rho + \partial_t j; \tag{9.48}$$

$$(-\partial_t^2 + \nabla^2) B = \text{curl } j. \tag{9.49}$$

Note that for simplicity I have again adopted units where the speed of light equals unity, and that we are now dealing with wave equations with sources.

- Schroedinger equation:

$$-i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = \left\{ -\frac{\hbar^2}{2m} \nabla^2 + V(t, \vec{x}) \right\} \psi(t, \vec{x}). \tag{9.50}$$

This particular PDE links the space and time dependence of the probability amplitude for finding a particle at a particular point. (Thankfully it is a linear PDE, which is why we can do such a lot with it.) This equation is very well understood and underlies much of humanity's quantum technology.

In terms of the generalized Euler PDE

$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} + c\, U_x + d\, U_y + e\, U + f = 0, \tag{9.51}$$

the Schroedinger equation corresponds to

$$a \to 0; \quad h \to 0; \quad b \to +\frac{\hbar^2}{2m}; \tag{9.52}$$

$$c \to -i\hbar; \quad d \to 0; \quad e \to -V(t, x); \quad f \to 0, \tag{9.53}$$

with the notational change $x \to t$, $y \to x$. There is a natural generalization from (1+1) to (2+1) and (3+1) dimensions.

- Continuity equation:
$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0. \tag{9.54}$$
Recall that this is a quasi-linear first order PDE. Because there are no second-order derivatives, the continuity equation cannot be put into Euler form.
- Euler equation (hydrodynamics):
$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho}. \tag{9.55}$$
Recall that this is a quasi-linear first order PDE. Because there are no second-order derivatives, the hydrodynamic Euler equation cannot be put into Euler form.

## 9.4 Quasi-linear Euler PDE

**Definition 14**
*The generalized quasi-linear Euler PDE is*

$$\begin{aligned} a(x,y,U,U_x,U_y)\, U_{xx} + 2h(x,y,U,U_x,U_y)\, U_{xy} + b(x,y,U,U_x,U_y)\, U_{yy} \\ = F(x,y,U,U_x,U_y), \end{aligned} \tag{9.56}$$

*where $a$, $h$, and $b$, are functions of $x$, $y$, $U$ and its first derivatives, and $F$ is a function of its indicated arguments. (And at least one of the second-order coefficients $a$, $b$, or $h$, is not identically zero.)*

- Note that the quasi-linear Euler equation is simply another name for the general quasi-linear second order PDE.

- Note that if you classify the quasi-linear Euler equations into elliptic, parabolic, hyperbolic by looking at the sign of $ab - h^2$, then the Euler type can depend not only on where you are in space, but also on the value of the dependent variable and its derivatives at that point.

## 9.5 Examples

Here are some examples of standard PDEs of considerable importance that fall under the heading of quasi-linear Euler type. (Though the interpretation might be considered a bit strained.)

- Navier–Stokes

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho} + \nu \nabla^2 \vec{v} \tag{9.57}$$

$$\vec{\nabla} \cdot \vec{v} = 0 \tag{9.58}$$

  - This is Euler's fluid dynamic equation (Newton's second law), plus incompressibility, plus conservation of mass, plus a particular model for viscosity.
  - Because of the viscosity term there is now at least one second-order term in the PDE — and because this second-order derivative occurs linearly the first of the two PDEs can be viewed as a quasi-linear Euler PDE.
  - Indeed this is a parabolic PDE.
  - These equations look innocent; they are very difficult to analyze.
  - The fact that they are nonlinear in the velocity field $\vec{v}$ is the ultimate source of all the difficulty.
  - Remember I told you that EUS is extremely difficult to prove for generic PDEs?
  - There is currently a US$1,000,000 Millennium prize from the Clay Mathematics institute for "substantial progress towards proving existence and smoothness" of the solutions:

    > Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier–Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to

make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

For the details of the challenge, see:
http://www.claymath.org/prizeproblems/navier_stokes.pdf

Please do *not* present me with any prize claims; see the rules as given on the website.

## 9.6 Exercises

Classify the following PDEs according to whether or not they are

- Euler (simple, constant coefficient).
- Euler (generalized, constant coefficient).
- Euler (variable coefficient, possibly with nonlinear source).
- Euler (quasi-linear).
- Non-Euler.

Whenever they fall into one of the many Euler classes above, further classify them according to whether they are elliptic, parabolic, hyperbolic.

(For some of these PDEs it will simply be a matter of reading the notes and copying the answers I've already given.)

a. $V^2\, V_{xy} + V_x\, V_y + (x^2 - y^2)V = 3xy.$
b. $U_{xxz} - 2(x+z)U_{xyz} - U_{xx} + \sin(xyz)U_{xx} = \cos(U)$
c. $U_t - UU_{xx} + 12xU_x = U.$
d. $Y_{xxx} - \cos Y = Y_t.$
e. $V_{xt} - \sin V = \exp(x+t).$
f. $Y_{xx} + \cos(xy)Y_{yxy} = Y + \ln(x^2 + y^3).$
g. $U_t = U_{xx} - 12U\, U_x.$
h. $V_{yx} + V_x + V_y = V_{xyy}.$
i. $U_{tt} - \cos(U_x) = U.$
j. $\cos x \cdot U_x + \sin t \cdot U_t = U.$
k. Schrodinger equation (with potential):

$$-i\partial_t \psi = \frac{1}{2m}\nabla^2\psi + V(x)\psi. \tag{9.59}$$

l. Monge–Ampere equation (2 variable):

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y). \tag{9.60}$$

m. Monge–Ampere equation (multi-variable):

$$\det \left[ \frac{\partial^2 u}{\partial x^i \, \partial x^j} \right] = f \left( x^i, u, \frac{\partial u}{\partial x^i} \right). \tag{9.61}$$

n. Navier–Stokes equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v}. \tag{9.62}$$

o. Tricomi equation:

$$y \, U_{xx} + U_{yy} = 0. \tag{9.63}$$

p. Frobenius–Mayer equation (special case, one dependent variable):

$$\frac{\partial U}{\partial x^i} = F_i(x, U). \tag{9.64}$$

q. Biharmonic equation:

$$\nabla^4 \Psi = 0. \tag{9.65}$$

That is, $(\nabla^2)^2 \, \Psi = 0$, or more explicitly:

$$\left[ \partial_x^2 + \partial_y^2 + \partial_z^2 \right]^2 \Psi = 0. \tag{9.66}$$

r. Benjamin–Bona–Mahony equation:

$$u_t + u_x + u u_x - u_{xxt} = 0. \tag{9.67}$$

s. Chaplygin equation:

$$u_{xx} + \frac{c^2 \, y^2}{c^2 - y^2} \, u_{yy} + y \, u_y = 0. \tag{9.68}$$

t. Boissinesq equation:

$$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}. \tag{9.69}$$

u. Euler–Darboux equation:

$$u_{xy} + \frac{\alpha\, u_x - \beta\, u_y}{x - y} = 0. \tag{9.70}$$

v. Korteweg–deVries–Burger:

$$u_t + 2uu_x - \nu\, u_{xx} + \mu\, u_{xxx} = 0. \tag{9.71}$$

w. Kirchever–Novikov equation:

$$\frac{u_t}{u_x} = \frac{1}{4}\frac{u_{xxx}}{u_x} - \frac{3}{8}\frac{u_{xx}^2}{u_x^2} + \frac{3}{8}\frac{4u^3 - g_2 u - g_3}{u_x^2}. \tag{9.72}$$

(Start by simplifying this a little.)

x. Lin–Tsien equation:

$$2u_{tx} + u_x\, u_{xx} - u_{yy} = 0. \tag{9.73}$$

y. Monge–Ampere equation (generalized):

$$\begin{aligned} &E(x,y,U,U_x,U_y)\left[U_{xx}U_{yy} - U_{xy}^2\right] \\ &+A(x,y,U,U_x,U_y)\, U_{xx} + B(x,y,U,U_x,U_y)\, U_{xy} + C(x,y,U,U_x,U_y)\, U_{yy} \\ &\qquad +D(x,y,U,U_x,U_y) = 0 \end{aligned}$$

or even more generally (multi variable case):

$$E(x^i, U, \partial_i U)\ \det\left[\frac{\partial^2 u}{\partial x^i\, \partial x^j}\right] + \sum_{ij} A^{ij}(x^i, U, \partial_i U)\, U_{,ij} + D(x^i, U, \partial_i U) = 0. \tag{9.74}$$

z. Cauchy–Riemann system of PDEs:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \tag{9.75}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{9.76}$$

After answering the question for the Cauchy–Riemann system itself, iterate these Cauchy–Riemann equations to find a pair of PDEs that decouple — they depend only on $u$, and only on $v$, but not both.

# Chapter 10

# Separation of Variables

The method of Separation of Variables (SOV) is a very general technique of fundamental importance for solving PDEs — I'll introduce it by looking at a specific example.

## 10.1 Sample problem

**Example used for illustration:**

Consider the wave equation

$$U_{xx} - U_{tt} = 0, \tag{10.1}$$

with boundary conditions

$$\begin{aligned}
U(x,0) &= f(x) \qquad \text{( the initial shape of the string )} && (10.2)\\
U_t(x,0) &= g(x) \qquad \text{( the initial velocity of the string )} && (10.3)\\
U(0,t) &= 0 \qquad \text{( pinned endpoint )} && (10.4)\\
U(L,t) &= 0 \qquad \text{( pinned endpoint )} && (10.5)
\end{aligned}$$

This is a linear PDE.

The last two boundary conditions are called homogeneous because they involve the dependent variable $U$ and its derivative linearly and homogeneously.

The first two boundary conditions are inhomogeneous.

## 10.2 The method

1. Use a trial solution of the variable-separated form:

$$U(x,t) = X(x)\,T(t). \tag{10.6}$$

In the case of the example we have chosen we find

$$U_{xx} = X''(x)\,T(t), \tag{10.7}$$

$$U_{tt} = X(x)\,T''(t), \tag{10.8}$$

and so

$$X''(x)\,T(t) - X(x)\,T''(t) = 0, \tag{10.9}$$

where $'$ stands for a derivative of the function with respect to its argument — either $x$ or $t$ as appropriate.

2. Separate the variables:

With luck, the PDE will allow you to gather all terms involving one independent variable on the left ($x$, say) and all other independent variables ($t$ in this case) on the right hand side.

We have

$$X''(x)\,T(t) = X(x)\,T''(t). \tag{10.10}$$

Dividing both sides by $X(x)\,T(t)$, we find

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}. \tag{10.11}$$

**Warning 7**
*If it turns out that you cannot separate the variables this way, then you will have to solve the DE some other way, because SOV won't work.*

**Warning 8**
*Though rare, sometimes you might have to try* additive *SOV*

$$U(x,t) = X(x) + T(t). \tag{10.12}$$

3. Find corresponding ODEs:

Each of the separated terms must be a constant.
Thus you find two ODEs to solve, involving as yet arbitrary constants.

Use the fact that the only way you can have a function $F(x)$, of one variable $x$, equal to another function $G(t)$, of another independent variable $t$, is to have both functions constant. Using this find ODEs that the separated functions must satisfy.

We have therefore:

$$\frac{X''(x)}{X(x)} = k, \qquad \text{and} \qquad \frac{T''(t)}{T(t)} = k, \tag{10.13}$$

where $k$ is some constant (as yet to be determined).

Thus we obtain a pair of ODEs for the unknown functions $X$ and $T$.

$$X'' = k\ X; \tag{10.14}$$

$$T'' = k\ T. \tag{10.15}$$

If there are more than two independent variables, you will need to continue the separation of variables procedure.

At the end, you should finish up with a collection of ODEs, one for each of the assumed functions in the separated variable form.

4. The ODEs can be solved in the usual way.

Doing so will give the functions $X(x)$ and $T(t)$ in terms of a selection of arbitrary constants.

$$X = A\ e^{\sqrt{k}x} + B\ e^{-\sqrt{k}x}; \tag{10.16}$$

$$T(t) = C\ e^{\sqrt{k}t} + D\ e^{-\sqrt{k}t}. \tag{10.17}$$

5. Apply homogeneous boundary conditions.

Now apply any homogeneous boundary conditions that you may have, in order to find out some information about the constants $A$, $B$, $C$, $D$ and $k$ that are lying about.

Note that, if $U(x,t) = X(x)\,T(t)$, and if one of the boundary conditions is $U(0,t) = 0 = X(0)T(t)$, then we must have $X(0) = 0$, since we certainly do not want $T(t) = 0$.

(That would make $U(x,t)$ identically zero, which is uninteresting.)

More specifically: If $T(t) = 0$, then $U(x,t) = 0$ for all values of $t$ and $x$, so we have the trivial solution [which obviously will not satisfy the remaining initial conditions, which have $U(x,t)$ nonzero for some values of $x$ and $t$]. Thus, we will always look for "non-trivial" solutions only.

*At this point it becomes apparent that some values of $k$ are acceptable, and others not.*

Indeed, in the case we are treating here, $k$ cannot in fact be positive:

- If $k > 0$, so we can write $k = b^2$ for some real $b$, then

$$X(x) = A\,e^{bx} + B\,e^{-bx}. \tag{10.18}$$

Then the boundary conditions imply

$$X(0) = 0 \quad : \quad A + B = 0, \tag{10.19}$$

$$X(L) = 0 \quad : \quad A\,e^{bL} + B\,e^{-bL} = 0, \tag{10.20}$$

which has the unique solution $A = 0 = B$.

But this solution would imply that $X(x) = 0$ for all $x$, and hence $U(x,t) = 0$ for all $x$ and $t$ — i.e. the solution is trivial.

*Thus, to avoid triviality, $k$ must be zero or negative*, and we can write

$$k = -b^2, \tag{10.21}$$

where $b$ is real or zero, to stress this fact.

- In that case, the solution to the equation for $X$ is, in general,

$$X(x) = A\ \sin(bx) + B\ \cos(bx). \tag{10.22}$$

Applying the conditions then gives

$$B = 0 \qquad \text{and} \qquad A\ \sin(bL) = 0. \tag{10.23}$$

Since we want to avoid trivial solutions, (so we don't want both $A$ and $B$ to be zero), we must ask that

$$\sin(bL) = 0, \tag{10.24}$$

which means that

$$bL = n\pi \qquad \text{where } n = 1, 2, 3, \ldots \text{ is a positive integer.} \tag{10.25}$$

$$i.e. \qquad b = \frac{n\pi}{L}. \tag{10.26}$$

- But this now means that $T(t)$ is very tightly constrained, it must satisfy

$$\frac{T''(t)}{T(t)} = k = -b^2 = -\frac{n^2\pi^2}{L^2}, \tag{10.27}$$

with general solution

$$T(t) = A_n\ \sin\left(\frac{n\pi t}{L}\right) + B_n\ \cos\left(\frac{n\pi t}{L}\right). \tag{10.28}$$

Thus we have discovered that

$$\begin{aligned} U(x,t) &= X(x)\,T(t) \qquad (10.29)\\ &= A_n\ \sin\left(\frac{n\pi x}{L}\right)\ \sin\left(\frac{n\pi t}{L}\right) + B_n\ \sin\left(\frac{n\pi x}{L}\right)\ \cos\left(\frac{n\pi t}{L}\right) \end{aligned}$$

is a solution of the wave equation *whatever* the constants $A_n$ and $B_n$, and *whatever* the integer $n$. Here we have written the constants $A$ and $B$ suffixed with $n$, to stress the fact that they can be *different* constants for each choice of $k = -b^2 = -n^2\pi^2/L^2$ , or equivalently, any choice of the integer $n$.

6. Superposition:

By the principle of superposition, *any arbitrary linear combination of these solutions is also a solution* satisfying the same homogeneous conditions.

Note that this works because the equation is linear and the boundary conditions are homogeneous (i.e., if we had done the above work and found a class of solutions satisfying non-homogeneous conditions, then we could not assert that arbitrary linear combinations of them also satisfy both the equation and the conditions!)

That is,

$$U(x,t) = \sum_{n=0}^{\infty} \left\{ A_n \ \sin\left(\frac{n\pi x}{L}\right) \ \sin\left(\frac{n\pi t}{L}\right) + B_n \ \sin\left(\frac{n\pi x}{L}\right) \ \cos\left(\frac{n\pi t}{L}\right) \right\} \tag{10.30}$$

is a solution too, satisfying the same homogeneous conditions.

**Warning 9**
*Note that the whole SOV approach is primarily designed for use on* linear *PDEs.*

*If the PDE is not linear, then even if you succeed in separating variables (difficult at best), you could not now appeal to superposition to construct the general solution.*

*Though there is a considerable industry of applying SOV to quasi-linear and nonlinear PDEs, that industry is aimed more at finding specific solutions rather than general solutions.*

*See, for instance, the Polyanin series of handbooks on PDEs for more than you ever wanted to know about "functional separation of variables".*

**Note 1**
*Thankfully, many of the most important PDEs* are *linear.*
*For example:*

- *wave equation;*
- *heat equation;*
- *Laplace's equation;*
- *many Euler equations;*
- *and their cousins.*

**Note 2**
*For more complicated nonlinear PDEs such as*

- *Einstein's equations of general relativity,*
- *Navier–Stokes equations of fluid mechanics,*

*the situation is* much *messier.*
*(And SOV is typically* not *a useful technique for solving those PDEs — unless you have an awful lot of symmetry in the problem; which means [by definition] that you cannot be looking for a truly general solution.)*

**Question 9**
*The generalization to non-homogeneous boundary conditions is actually not too difficult. (Just make sure it's a linear PDE.) Any ideas?*

> 7. Fit series:
>
> Now try to fit the series solution you have found to the remaining non-homogeneous conditions. (Typically, but not always, initial conditions.)

In general you will get something like the Fourier problem, for which the solution is well known. (You have not seen Fourier series yet. Fourier series are the next topic.)

The condition

$$U(x, 0) = f(x), \tag{10.31}$$

gives

$$\sum_{n=0}^{\infty} B_n \sin(n\pi x/L) = f(x), \qquad (1) \tag{10.32}$$

(note that there is effectively no $B_0$) and the condition

$$U_t(x, 0) = g(x), \tag{10.33}$$

gives

$$\sum_{n=0}^{\infty} A_n \frac{n\pi}{L} \sin(n\pi x/L) = g(x). \qquad (2) \tag{10.34}$$

*Historically this is the way Fourier series were first encountered.*

Since $f(x)$ and $g(x)$ above are arbitrary functions of $x$, *and* physically we know the vibrating string had better have a mathematical solution for arbitrary initial data, *and* the SOV technique seems to indicate that $f(x)$ and $g(x)$ must be sums of sines and cosines, this *strongly suggests* that (more or less) arbitrary functions of $x$ can be represented as sums of sines and cosines.

This is the "miracle" of Fourier series, and at first mathematicians and physicists simply did not believe their own results.

From Fourier Series theory the constants $A_n$ and $B_n$ can be found in the usual way. (I will justify these formulae later.)
From (1)

$$\boxed{A_n = \frac{2}{n\pi} \int_0^L g(x) \sin(n\pi x/L) \, \mathrm{d}x \qquad [n = 1, 2, 3, ....],} \tag{10.35}$$

and from (2) we have

$$\boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L) \, \mathrm{d}x \qquad [n = 1, 2, 3, ....],} \tag{10.36}$$

and so the general solution satisfying all the given conditions is

$$\boxed{U(x,t) = \sum_{n=0}^{\infty} \left[A_n \sin(n\pi x/L) \sin(n\pi t/L) + B_n \sin(n\pi x/L) \cos(n\pi t/L)\right],} \tag{10.37}$$

with the specific $A_n$, $B_n$ found above.

8. A specific example:

Suppose the string has length $L = 2$, and was plucked in such a way that

$$f(x) = \begin{cases} x/10 & \text{for } 0 = x = 1; \\ \\ (2-x)/10 & \text{for } 1 = x = 2. \end{cases} \tag{10.38}$$

and

$$g(x) = 0, \tag{10.39}$$

corresponding to the string being initially held fixed.

Then you find

$$A_n = 0 \qquad \text{for all } n, \tag{10.40}$$

and

$$B_n = \frac{2}{n\pi}\,[1 - \cos(n\pi)] \left[\frac{2}{n\pi}\,\sin(n\pi/2) - \cos(n\pi/2)\right], \tag{10.41}$$

and the explicit solution to our example is

$$\boxed{\begin{aligned} U(x,t) \;&=\; \sum_{n=0}^{\infty} \left\{ \frac{2}{n\pi}\,[1-\cos(n\pi)]\,\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \cos\left(\frac{n\pi}{2}\right)\right] \right. \\ &\qquad \left. \times \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right) \right\}, \end{aligned}} \tag{10.42}$$

which is perhaps not very edifying!

You can use Maple to plot a diagram of the sum taking, say, the first ten terms (this should give a pretty good picture of the situation).

**Exercise 2** *Use* Maple *to generate a plot, truncated at the 30'th term, for time values* $t$ = 0.0,0.5, 1.0, 1.5, *and* 2.0. *You should see quite clearly the development of the oscillations in the string. Note that the rounded edges near peaks are simply due to the truncation.*

## 10.3 Comments on SOV

### 10.3.1 Possible complications

I have presented only the simplest form of the SOV technique. It can be modified in much more general ways. For example, for systems where there is some version of spherical symmetry it is often useful to write

$$U(t, x, y, z) = T(t)\ R(r)\ L(\theta)\ \Phi(\phi). \tag{10.43}$$

If you then consider the $(3 + 1)$–dimensional wave equation it will separate, but you might be a little surprised at the results

- $T(t)$ is a complex exponential (cis: cosine plus $i$ sine).
- $\Phi(\phi)$ is a complex exponential (cis: cosine plus $i$ sine).
- $L(\theta)$ is a Legendre polynomial in the variable $\cos\theta$.
- $R(r)$ is a spherical Bessel function.
- The combinations $Y(\theta, \phi) = L(\theta)\ \Phi(\phi)$ are the spherical harmonics.

In other words, sines and cosines often arise in SOV, but more complicated functions also show up.

If you are solving the wave equation on a circular drum-head [two space dimensions, one time dimension] you will typically get

$$U(t, x, y) = T(t)\ B(r)\ \Phi(\phi). \tag{10.44}$$

Here:

- $T(t)$ is a complex exponential (cis: cosine plus $i$ sine).
- $\Phi(\phi)$ is a complex exponential (cis: cosine plus $i$ sine).
- $B(r)$ is an ordinary Bessel function.

Again, sines and cosines often arise in SOV, but more complicated functions also show up.

**Comment 5** *Of course, the fact that Bessel functions (both ordinary and spherical) show up in such simple applications of SOV to the wave equation is the fundamental reason why applied mathematicians are so interested in Bessel functions — this is why they have been given a special name, and why the properties of Bessel functions have been so intensely investigated.*

### 10.3.2 A sufficient condition

A sufficient condition for the SOV technique to work is described below:

Consider the linear homogeneous PDE

$$D_n U_n - \lambda_n U_n = 0, \tag{10.45}$$

where $D_n$ is some partial differential operator in $n$ independent variables.

*If* you can find a coordinate system such that the partial differential operator decomposes into something proportional to a sum of an ordinary differential operator plus a lower-dimensional partial differential operator,

$$D_n = D_1 + h_1(x_1)\ D_{n-1}, \tag{10.46}$$

where $D_1$ is an ordinary differential operator which involves only one of the independent variables, $h_1$ is a function which depends only on $x_1$, and $D_{n-1}$ involves the remaining $n-1$ independent variables, *then* you can begin to apply the SOV technique.

**Definition 15 Partially separable coordinates:**
*A coordinate system such that*

$$D_n = D_1 + h_1\ D_{n-1}, \tag{10.47}$$

*is said to be partially separable for the partial differential operator $D$.*

To see how this works take

$$U_n = X(x_1)\ U_{n-1}(x_{i\neq 1}), \tag{10.48}$$

then

$$D_n U_n = (D_1 X)\ U_{n-1} + X\ h_1\ (D_{n-1} U_{n-1}), \tag{10.49}$$

which implies

$$\frac{D_1 X}{X} + h_1\ \frac{D_{n-1} U_{n-1}}{U_{n-1}} - \lambda_n = 0. \tag{10.50}$$

That is

$$\left(\frac{D_1 X}{X} - \lambda_n\right) \frac{1}{h_1} + \frac{D_{n-1} U_{n-1}}{U_{n-1}} = 0, \tag{10.51}$$

with the first term depending only on $x_1$ and the second only on the other $n-1$ independent variables $x_{i\neq 1}$.

Therefore there exists a number $\lambda_{n-1}$, the separation constant, such that

$$D_{n-1} U_{n-1} = \lambda_{n-1}\, U_{n-1}; \tag{10.52}$$

$$D_1 X = \{\lambda_n - \lambda_{n-1}\, h_1\}\, X. \tag{10.53}$$

This has reduced the number of independent variables by one, and split the problem into a simpler PDE in $n-1$ independent variables, plus a linear ODE. If you can iterate this process all the way down to $n = 1$ then the system is completely separable. (And even if the problem is only partially separable, that may still represent significant progress.)

### 10.3.3 Examples

For the Laplacian operator the following coordinate systems are separable:

- Cartesian coordinates

$$\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2. \tag{10.54}$$

- Spherical polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2} \tag{10.55}$$

  Now apply the above analysis to re-write this as:

$$\nabla^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\left[\left\{\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta}\right\} + \frac{1}{\sin^2\theta}\left\{\frac{\partial^2}{\partial \phi^2}\right\}\right] \tag{10.56}$$

  Note it is completely separable.

- There are nine other separable coordinate systems known for the 3-d Laplacian; some are relatively simple (cylindrical polar coordinates) others are more obscure (prolate and oblate spheroidal coordinates). And some I've never heard of.

### 10.3.4 Boundary Conditions

To finish applying SOV you will need to verify that in this same coordinate system the boundary conditions "factorize" in the sense that they can be written as independent sets of boundary conditions for each variable $x_i$ that do not "cross communicate" with each other.

## 10.4 Exercises

### 10.4.1 Heat equation

A square copper sheet has its edges maintained at prescribed temperatures. Along the $x$ and $y$ axes the temperature is held to zero (say by a nice big block of ice). The temperature is also held to zero along the side given by the line $x = 1$. Finally, along the fourth edge of the square at $y = 1$ the temperature is held at $100x(1-x)$ — so that it is zero at the edges and rises quadratically with a maximum of 25 at the centre of this edge.
When all has settled down to equilibrium, the distribution of heat in the slab satisfies Laplace's equation

$$U_{xx} + U_{yy} = 0 \qquad (0 \leq x, y \leq 1). \tag{10.57}$$

where $U(x, y)$ is the temperature at the point $(x, y)$ in the slab.

From the situation described, we have the boundary conditions:

$$U(x, 0) = 0; \tag{10.58}$$

$$U(0, y) = 0; \tag{10.59}$$

$$U(1, y) = 0; \tag{10.60}$$

$$U(x, 1) = 100x(1 - x). \tag{10.61}$$

a. Find the distribution of temperature in the slab at equilibrium.

   [Of course, you will try separation of variables: $U = X(x)\, Y(y)$, and deduce that $X'' = -b^2X$ and $Y'' = b^2Y$ where b is real. You will also demonstrate that X cannot satisfy the alternative possibility $X'' = +b^2X$ for real $b$. Then you will apply the homogeneous BC to find out about $b$.

b. An isotherm is a curve of constant temperature. Sketch the isotherms for temperatures 0, 10, 20, 30 and 40 degrees. [Here is your chance to use Maple.]

### 10.4.2 Applying Laplace's general solution

Have a go at the previous question by noting that the general solution of $U_{xx} + U_{yy} = 0$ is given by

$$U(x, y) = F(x + iy) + G(x - iy), \tag{10.62}$$

(where $F$ and $G$ are arbitrary functions) and trying to fit this general solution to the given conditions.

*Do not be surprised to find that it seems impossible.*

### 10.4.3 Elastic string (from SOV to d'Alembert and back again)

For a finite elastic string stretched between $x = 0$ and $x = L$, the equation describing its displacement $U(x, t)$ away from the equilibrium configuration at position $x$ at time $t$ is the wave equation

$$\frac{\partial^2 U(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 U(x,t)}{\partial t^2} = 0. \tag{10.63}$$

Here $c$ is a constant depending on the elastic properties of the string and its tension. We shall suppose $c = 1$. The appropriate boundary conditions for the problem are:

i. BC1: $U(x, 0) = f(x)$ for $0 < x < L$, describing the initial shape of the string when first plucked.
ii. BC2: $U_t(x, 0) = 0$ for $0 < x < L$, stating that the string initially was held in the shape of $f(x)$ and was then released from rest.
iii. BC3: $U(0, t) = 0$, stating that the string is permanently fixed at $x = 0$.
iv. BC4: $U(L, t) = 0$, stating that the string is also permanently fixed at $x = L$.

When you solve this equation using the method Separation of Variables, you find the solution is of the form:

$$U(x, t) = \sum_{n=0}^{\infty} A_n \ \sin(n\pi x/L) \ \cos(n\pi t/L). \tag{10.64}$$

Here

$$A_n = \frac{2}{L} \int_0^L f(x) \ \sin(n\pi x/L) \ \mathrm{d}x. \tag{10.65}$$

You will notice that the solution you have here is defined for all $x$ and $t$. For any fixed $t$ it is an odd function for all $x$ which is periodic with period $2L$ and for any fixed $x$ it is an even function which is periodic with period $2L$.

On the other hand, you know that the general solution to the wave equation is $U(x,t) = F(x+t) + G(x-t)$ where $F$ and $G$ are arbitrary. The boundary conditions then imply that:

$$F(x) + G(x) = f(x) \text{ for } 0 < x < L; \tag{10.66}$$

$$F'(x) + G'(x) = 0 \text{ for } 0 < x < L; \tag{10.67}$$

$$F(t) + G(-t) = 0 \text{ for all } t; \tag{10.68}$$

$$F(L+t) + G(L-t) = 0 \text{ for all } t. \tag{10.69}$$

Use these conditions to show that $U(x,t)$ has the properties alluded to above, viz that it is defined for all $x$ and $t$, for any fixed $t$ it is an odd function for all $x$ which is periodic with period $2L$ and for any fixed $x$ it is an even function which is periodic with period $2L$.

Hence show in general that the solution can be expressed in the form you found using separation of variables.

### 10.4.4 Heat equation

Solve the heat equation for diffusion of heat down a bar of length $L = 10$:

$$U_{xx} = \frac{1}{k^2} \, U_t \tag{10.70}$$

subject to the conditions

$$\begin{aligned} U(x,0) &= x \text{ for } 0 < x < 5 && (10.71) \\ &= 10 - x \text{ for } 5 < x < 10 && (10.72) \\ U(0,t) &= 0 = U(10,t). && (10.73) \end{aligned}$$

Take $k = 1$ for argument's sake.

Graph the distribution of temperature down the bar:

i. Initially.
ii. At time $t = 3$ (Plot only the first few terms of the Fourier series you should have. Indeed, if you are a Maple fanatic, you could present a rather good time sequence here. Choose a value for $k$ for yourself.)
iii. After an extremely long time.

# Chapter 11

# Fourier series

## 11.1 Basics

Based on the example we used to describe the SOV principle, we found strong reasons for suspecting that relatively general functions $f(x)$ should be representable as sums of sines and cosines:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \left[A_n \, \cos(\pi n x/L) + B_n \, \sin(\pi n x/L)\right]. \qquad (11.1)}$$

In this chapter we will ask (and partially answer) how general this sort of decomposition is, and how to calculate the coefficients $A_n$ and $B_n$.

## 11.2 Fourier coefficients

As it turns out, calculating the coefficients is easy: Suppose we have a function $f(x)$ defined on the interval $(0, L)$, and suppose that in that interval it is described by a Fourier series

$$f(x) = \sum_{n=0}^{\infty} \left[A_n \, \cos(\pi n x/L) + B_n \, \sin(\pi n x/L)\right], \qquad (11.2)$$

which we shall assume converges (at least "almost everywhere", in some point-wise sense).

**Warning 10** *There is nothing sacred about the use of the interval* $[0, L]$*. Any interval* $[a, b]$ *could be used as long as you are willing to translate and rescale the domain of the function. You could, for instance always choose to work on the domain* $[0, 1]$*. Working on* $[0, L]$ *is a compromise between complete generality and obtaining tractable equations.*

Note that the Fourier sum is automatically periodic under $x \to x + 2L$, *even if the original function* $f(x)$ *is undefined outside of this range.*

Now consider the four integrals:

$$\int_{-L}^{L} \cos(\pi n x/L)\ \cos(\pi m x/L)\ \mathrm{d}x = L\,(\delta_{mn} + \delta_{m0}\,\delta_{n0})\,; \tag{11.3}$$

$$\int_{-L}^{L} \sin(\pi n x/L)\ \sin(\pi m x/L)\ \mathrm{d}x = L\,(\delta_{mn} + \delta_{m0}\,\delta_{n0})\,; \tag{11.4}$$

$$\int_{-L}^{L} \sin(\pi n x/L)\ \cos(\pi m x/L)\ \mathrm{d}x = 0; \tag{11.5}$$

$$\int_{-L}^{L} \cos(\pi n x/L)\ \sin(\pi m x/L)\ \mathrm{d}x = 0. \tag{11.6}$$

**Proof:**
For example, suppose to start with that both $n + m$ and $n - m$ are nonzero. Then, taking $x = z\,L$ and $dx = L\,dz$, we have

$$\int_{-L}^{L} \cos(\pi n x/L)\ \cos(\pi m x/L)\ \mathrm{d}x = L \int_{-1}^{1} \cos(\pi n z)\ \cos(\pi m z)\ \mathrm{d}z \tag{11.7}$$

$$= \frac{L}{2} \int_{-1}^{1} \{\cos(\pi[n+m]z) + \cos(\pi[n-m]z)\}\ \mathrm{d}z \tag{11.8}$$

$$= \frac{L}{2}\,\frac{2}{\pi} \left\{ \frac{1}{n+m} \sin(\pi[n+m]z)|_{-1}^{1} + \frac{1}{n-m} \sin(\pi[n-m]z)|_{-1}^{1} \right\} \tag{11.9}$$

$$= 0. \tag{11.10}$$

Thus this integral is definitely zero if both $n + m$ and $n - m$ are nonzero.

If $n+m=0$ but $n-m\neq 0$ (i.e., $n=-m\neq 0$) then

$$\begin{aligned}\int_{-L}^{L}\cos(\pi nx/L)\ \cos(\pi mx/L)\ \mathrm{d}x &= L\int_{-1}^{1}\cos(\pi nz)\ \cos(\pi mz)\ \mathrm{d}z \\ &= L\ \int_{-1}^{1}\cos^2(\pi nz)\ \mathrm{d}z && (11.11)\\ &= L\times 2\times\frac{1}{2} && (11.12)\\ &= L && (11.13)\end{aligned}$$

Similarly if $n-m=0$ but $n+m\neq 0$ (i.e., $n=m\neq 0$) then

$$\begin{aligned}\int_{-L}^{L}\cos(\pi nx/L)\ \cos(\pi mx/L)\ \mathrm{d}x &= L\int_{-1}^{1}\cos(\pi nz)\ \cos(\pi mz)\ \mathrm{d}z \\ &= L\ \int_{-1}^{1}\cos^2(\pi nz)\ \mathrm{d}z && (11.14)\\ &= L\times 2\times\frac{1}{2} && (11.15)\\ &= L && (11.16)\end{aligned}$$

Finally if $n=m=0$

$$\begin{aligned}\int_{-L}^{L}\cos(\pi nx/L)\cos(\pi mx/L)\ \mathrm{d}x &= \int_{-L}^{L} 1\ \mathrm{d}x && (11.17)\\ & && (11.18)\\ &= 2L. && (11.19)\end{aligned}$$

Collecting these results, in all cases we have

$$\int_{-L}^{L}\cos(\pi nx/L)\ \cos(\pi mx/L)\ \mathrm{d}x = L\ \left(\delta_{mn}+\delta_{m0}\,\delta_{n0}\right). \tag{11.20}$$

The other three integrals are just minor variations on this theme.

**Exercise 3** *Check the other three integrals.*

So now we play a trick. Take $f(x)$ to be defined in $x \in (0, L)$ and extend it, *in any completely arbitrary way*, to a function $\hat{f}(x)$ defined on $x \in [-L, +L]$.

**Warning 11** *There is again nothing sacred about the use of the interval* $[-L, L]$*. Any interval* $[a, b]$ *could be used as long as you are willing to translate and rescale the domain of the function. You could, for instance, always choose to work on the domain* $[-1, 1]$*. Working on* $[-L, L]$ *is a compromise between complete generality and obtaining tractable equations.*

Assume that $\hat{f}(x)$, defined on $[-L, L]$, possesses a Fourier series

$$\hat{f}(x) = \sum_{n=0}^{\infty} \left[A_n \cos(\pi n x/L) + B_n \sin(\pi n x/L)\right]. \tag{11.21}$$

Now multiply both sides of this equation by $\cos(\pi m x/L)$ and integrate from $-L$ to $+L$.

$$\int_{-L}^{L} \cos(\pi m x/L)\ \hat{f}(x)\ \mathrm{d}x = \sum_{n=0}^{\infty} \left[A_n\ L\ \left(\delta_{mn} + \delta_{m0}\ \delta_{n0}\right)\right]. \tag{11.22}$$

Then the sum over $n$ is trivially done and

$$\boxed{\begin{aligned} A_0 &= \frac{1}{2L} \int_{-L}^{L} \hat{f}(x)\ \mathrm{d}x. && (11.23) \\ A_{n\neq 0} &= \frac{1}{L} \int_{-L}^{L} \cos(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x. && (11.24) \end{aligned}}$$

Similarly if we multiply both sides of this equation by $\sin(\pi m x/L)$ and integrate from $-L$ to $+L$ we have

$$\int_{-L}^{L} \sin(\pi m x/L)\ \hat{f}(x)\ \mathrm{d}x = \sum_{n=0}^{\infty} \left[B_n\ L\ \left(\delta_{mn} + \delta_{m0}\ \delta_{n0}\right)\right]. \tag{11.25}$$

Then, summing over $n$, we have

$$\boxed{\begin{aligned} B_0 &= 0. && (11.26) \\ B_{n\neq 0} &= \frac{1}{L} \int_{-L}^{L} \sin(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x. && (11.27) \end{aligned}}$$

Currently, these formulae have been derived under the *assumption* that the series converges. We *assume* that

$$\boxed{\hat{f}(x) = \sum_{n=0}^{\infty} \left[ A_n \cos(\pi n x/L) + B_n \sin(\pi n x/L) \right], \qquad (11.28)}$$

with $\hat{f}(x)$ defined on $[-L, +L]$, makes sense!

**Remarks:**

- These formulae for the coefficients are often called the Euler–Fourier formulae. (Or sometimes just the Euler formulae — Euler did a truly tremendous amount of research on PDEs.)
- The above shows how to find $A_m$ and $B_m$, given that $f(x)$ is extended in some arbitrary way to $\hat{f}(x)$, and given that $\hat{f}(x)$ can be written as an infinite Fourier series.
- It does not (yet) follow that, if you were to calculate the $A_m$ and $B_m$ by this prescription, and put these values of $A_m$ and $B_m$ back into the series, that the resulting series would always converge to $f(x)$. (In fact it does not *always* converge; at best it is convergent "almost everywhere".)
- There is a large degree of arbitrariness in the prescription — $f(x)$ can be extended to $\hat{f}(x)$ in an arbitrary way and we still seem to get a sensible Fourier series?

  What on earth is going on here? [Explanation below.]
- A necessary condition for the Fourier series to exist is that the Fourier coefficients be well defined, which in turn requires (at the very least), that $\hat{f}(x)$ be integrable.
- Now let's try for sufficient conditions.

## 11.3 Fourier series

**Definition 16 Piecewise continuity:**
*A function $f(x)$ is piecewise continuous on the interval $a < x < b$ if the interval can be partitioned into a finite number of sub-intervals by using the points $a = x_0 < x_1 < x_2 ... < x_n = b$ and verifying that:*

- *$f(x)$ is continuous on each of the subintervals $(x_i, x_{i+1})$.*
- *$f(x)$ approaches a finite limit as the endpoints of each subinterval are approached from within the subinterval.*

  *That is, if*

  $$f(x_i^+) = \lim_{h \to 0^+} f(x_i + h) \qquad \text{and} \qquad f(x_i^-) = \lim_{h \to 0^-} f(x_i + h) \quad (11.29)$$

  *both exist and are finite for all $i = 0, 1, 2, \ldots n$.*

**Theorem 7 Fourier's general theorem:**
*Suppose that the functions $\hat{f}(x)$ and $\hat{f}'(x)$ are both piecewise continuous on the interval $-L \leq 0 \leq L$, then*

- *$\hat{f}(x)$ has a Fourier series whose coefficients are determined by the Euler–Fourier formulae above.*
- *the Fourier series converges to $\hat{f}(x)$ at all points where $\hat{f}(x)$ is continuous, and converges to $\frac{1}{2}[\hat{f}(x^+) + \hat{f}(x^-)]$ at points of discontinuity.*

**Remarks:**

- The conditions of this theorem are certainly sufficient for the convergence of the Fourier series. They are not necessary. Further, they are not even the most general sufficient conditions. *As far as I can tell, nobody knows the minimal necessary and sufficient conditions for the Fourier series to converge to the function almost everywhere.*

- That is, we know many different theorems specifying necessary conditions, and many other quite different theorems specifying sufficient conditions, but *no-one* seems to know the minimal necessary and sufficient conditions for convergence. Research is ongoing...

  For more background information, see:
  http://en.wikipedia.org/wiki/Convergence_of_Fourier_series

**Proof: Convergence of the Fourier series:**

We here reproduce Kreyszig's proof of convergence for the Fourier series for functions $\hat{f}(x)$ which are continuous, have continuous second derivatives, and which are periodic with period $2L$. This convergence theorem is useful because of its simplicity and because it illustrates the use of convergence theorems you should already have seen.

The more general case enunciated above, and the proof that it actually converges to the values stated, requires more analysis than we have done.

Note that under the conditions stated, $\hat{f}(-L) = \hat{f}(L)$ and $\hat{f}'(-L) = \hat{f}'(L)$. Integrating the Euler–Fourier formulae (for $n \neq 0$) by parts we find that

$$\begin{aligned} A_n &= \frac{1}{L} \int_{-L}^{L} \cos(\pi n x/L)\; \hat{f}(x)\; \mathrm{d}x \\ &= \left.\frac{f(x)\; \sin(\pi n x/L)}{n\pi}\right|_{-L}^{+L} - \frac{1}{n\pi} \int_{-L}^{L} \sin(\pi n x/L)\; \hat{f}'(x)\; \mathrm{d}x \\ &= -\frac{1}{n\pi} \int_{-L}^{L} \sin(\pi n x/L)\; \hat{f}'(x)\; \mathrm{d}x. \end{aligned} \tag{11.30}$$

(The contributions from upper and lower limits vanish because the sine function is zero there.)

Now integrate by parts again

$$A_n = \left.\frac{f'(x)\; \cos(\pi n x/L)}{n\pi\; (n\pi/L)}\right|_{-L}^{+L} - \frac{1}{n\pi\; (n\pi/L)} \int_{-L}^{L} \cos(\pi n x/L)\; \hat{f}''(x)\; \mathrm{d}x$$

$$= \quad -\frac{L}{n^2\pi^2} \int_{-L}^{L} \cos(\pi n x/L)\ \hat{f}''(x)\ \mathrm{d}x. \tag{11.31}$$

(The contributions from upper and lower limits cancel because the second derivative is periodic.)
But now, because $\hat{f}(x)$ by assumption has a continuous second derivative on $[-L, +L]$, it must be bounded

$$|f''(x)| < M. \tag{11.32}$$

Therefore

$$|A_n| < \frac{L}{n^2\pi^2} \int_{-L}^{L} |\cos(\pi n x/L)\ \hat{f}''(x)|\ \mathrm{d}x < \frac{L}{n^2\pi^2} \int_{-L}^{L} M \mathrm{d}x < \frac{2ML^2}{n^2\pi^2}. \tag{11.33}$$

Similarly, we can bound the $B_n$ for all $n$ (just repeat the analogous steps)

$$|B_n| < \frac{2ML^2}{n^2\pi^2}. \tag{11.34}$$

But then

$$|\text{Fourier series}| < |A_0| + 4M \left( 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right). \tag{11.35}$$

And this series definitely does converge. Therefore the Fourier series converges.

**Note 3** *It is a standard result that*

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} \dots = \zeta(2) = \frac{\pi^2}{6}. \tag{11.36}$$

*This is called the "Basel problem", and was solved by, surprise, Euler.*

**Note 4**
*(For the dedicated): In fact by the Weirstrauss uniform convergence test the Fourier series converges uniformly; which ultimately justifies the way we have cavalierly interchanged summations and integrations.*

**Note 5**
*(For the dedicated): A considerably more subtle proof is needed if you want to get away with piecewise continuity as your only input assumption.*

**Note 6**

$$|\text{Fourier series}| < |A_0| + \frac{2M\pi^2}{3}. \tag{11.37}$$

**Periodicity:**

- Since $\sin(\pi x/L)$ and $\cos(\pi x/L)$ are functions which are periodic with period $2L$, it follows that the Fourier series are themselves functions which are periodic with period $2L$. Thus, unless the function $\hat{f}(x)$ has the same period, the Fourier series and the function it is obtained from *can only agree on the original interval*.
- On the other hand, if $\hat{f}(x)$ has period $2L$ then the series and the function agree (almost) everywhere.

**References:**

- *Advanced Calculus*, pp 321 ff.
- Kreyszig, E. *Advanced Engineering Mathematics*, pp 581 ff.
- In fact, any text on Engineering Mathematics will probably have a discussion of Fourier series.

## 11.4 Fourier sine series

Now we are going to use the freedom of the extension process $f : [0, L] \to \hat{f} : [-L, L]$ to see if we can come up with simpler versions of the Fourier series.

Suppose we construct $\hat{f}(x)$ so that it is *odd* in the interval $[-L, L]$. That is:

$$\hat{f}(x) = f(x) \text{ for } x \in (0, L) \tag{11.38}$$
$$\hat{f}(x) = -f(-x) \text{ for } x \in (-L, 0) \tag{11.39}$$

Then in the Euler–Fourier formulae all the coefficients $A_n$ are zero and we have

$$\boxed{f(x) = \sum_{n=1}^{\infty} \left[ B_n \sin(\pi n x/L) \right],} \tag{11.40}$$

with

$$\boxed{B_n = \frac{1}{L} \int_{-L}^{L} \sin(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x = \frac{2}{L} \int_{0}^{L} \sin(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x.} \tag{11.41}$$

But then we can use the general Fourier theorem to obtain the more specific result below:

**Theorem 8 Fourier sine theorem:**
*If $f(x)$ is piecewise continuous, with piecewise continuous derivatives, then the Fourier sine series converges for all values of $x$ in the interval $[0, L]$.*

*Furthermore:*

*i. If $x$ is a point in $(0, L)$ where the function $f(x)$ is continuous, then the series converges to $f(x)$.*

*ii. If $x$ is a point in $(0, L)$ where $f(x)$ has a discontinuity, then the series converges to*

$$\frac{1}{2}[f(x^+) + f(x^-)]. \tag{11.42}$$

*iii. At the points $x = 0$ and $x = L$, the series converges to $y = 0$.*
*[Not to $f(0)$ and $f(L)$.]*

**Proof:**

The proof of the full theorem requires much more analysis than we have developed. However, there is a proof of convergence given in Kreyszig for $C^2$ functions which are periodic with period $2L$ which is relatively straightforward. We have reproduced it above for the full Fourier series case; and nothing extra is required for the Fourier sine theorem.

## 11.5 Fourier cosine series

As for the sine series:

Suppose we construct $\hat{f}(x)$ so that it is *even* in the interval $[-L, L]$. That is:

$$\hat{f}(x) = f(x) \text{ for } x \in (0, L); \tag{11.43}$$
$$\hat{f}(x) = +f(-x) \text{ for } x \in (-L, 0). \tag{11.44}$$

Then in the Euler–Fourier formulae all the coefficients $B_n$ are zero and we have

$$\boxed{f(x) = \sum_{n=0}^{\infty} \left[A_n \cos(\pi n x/L)\right],} \tag{11.45}$$

with

$$\boxed{A_n = \frac{1}{L}\int_{-L}^{L} \cos(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x = \frac{2}{L}\int_{0}^{L} \cos(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x;} \tag{11.46}$$

$$\boxed{A_0 = \frac{1}{2L}\int_{-L}^{L} \hat{f}(x)\ \mathrm{d}x = \frac{1}{L}\int_{0}^{L} \hat{f}(x)\ \mathrm{d}x.} \tag{11.47}$$

**Theorem 9 Fourier cosine theorem:**
*If $f(x)$ is piecewise continuous, with piecewise continuous derivatives, then the Fourier cosine series above converges for all values of $x$ in the interval $[0, L]$.*

*Furthermore:*

*i. If $x$ is a point in $(0, L)$ where $f(x)$ is continuous, then the series converges to $f(x)$.*

*ii. If $x$ is a point in $(0, L)$ where $f$ has a discontinuity, then the series converges to*

$$[f(x^+) + f(x^-)]/2. \tag{11.48}$$

*iii. At the points $x = 0$ and $x = L$, the series converges to $f(0)$ and $f(L)$ respectively.*

**Proof:** Again, as for the sine functions.

Note the full Fourier theorem is applied to $\hat{f}(x)$ in the interval $[-L, L]$; whereas the Fourier cosine theorem tells you about $f(x)$ in the interval $[0, L]$.

**Important Note:**
In the case of the Fourier cosine series, it is common practice to re-write the series as

$$f(x) = \frac{\bar{A}_0}{2} + \sum_{n=1}^{\infty} \left[\bar{A}_n \cos(\pi n x/L)\right], \tag{11.49}$$

with

$$\bar{A}_n = \frac{2}{L}\int_0^L \cos(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x, \tag{11.50}$$

where the *same* formula for $\bar{A}_n$ now holds for all $n = 0, 1, 2, 3, \dots$

This has the effect of simplifying the Euler formulae for the coefficients at the cost of putting an explicit 2 in the contribution of the $n = 0$ mode to the Fourier series.

Personally, if I were to bother doing this at all, I'd go one step further and define new coefficients $a_n$ such that

$$f(x) = \sum_{n=-\infty}^{\infty} a_n\ \cos(\pi n x/L), \tag{11.51}$$

with

$$a_n = \frac{1}{L}\int_0^L \cos(\pi n x/L)\ \hat{f}(x)\ \mathrm{d}x, \tag{11.52}$$

so that

$$a_{-n} = a_{+n}. \tag{11.53}$$

This gets rid of the explicit occurrence of the 2, completely. There's no longer any explicit 2's anywhere in either the Euler formula or the Fourier series — of course the 2 is now hiding implicitly in the fact that the summation runs from $-\infty$ to $+\infty$.

**Symmetry:**

- Since the $\sin(\pi x/L)$ are odd functions, it follows that the sine series is an odd function. Therefore, expressing $f(x)$ as a sine series can only be true for the interval $[0, L]$, unless of course $f(x)$ is itself odd, in which case the sine series agrees (as much as it can) with $f(x)$ over the entire interval $[-L, L]$.

- On the other hand, the $\cos(\pi x/L)$ are even functions, so a cosine series is an even function. Therefore, expressing $f(x)$ as a cosine series can only be true for the interval $[0, L]$, unless of course $f(x)$ is itself even, in which case the cosine series agrees (as much as it can) with $f(x)$ over the entire interval $[-L, L]$.

- If a function $f(x)$ is odd (even) then the full Fourier series for the function has only sine functions (cosine functions) in it. Thus we obtain the sine (cosine) series for a function $f(x)$ on $[0, L]$ if we extend $f(x)$ to the interval $[-L, L]$ as an odd (even) function $\hat{f}(x)$ and then take the full Fourier series for it.
- For this reason we really only needed to consider the full Fourier series above!!

## 11.6 Truncation errors: Gibbs phenomenon

- Naturally when plotting the Fourier series you will need to truncate. As you may surmise from the examples above, the error made in a truncation depends on the point $x$ (for instance, note that near jumps and sharp points in the function the series fluctuates rapidly and the error rises).
- Nevertheless, if you use the orthogonality properties of the Fourier series, then you can estimate the size of the error.
- See next chapter...

## 11.7 Examples of Fourier series

- A Fourier sine series for

$$f(x) = \begin{cases} x & \text{for } 0 < x < 1; \\ (2-x) & \text{for } 1 < x < 2. \end{cases} \tag{11.54}$$

The coefficients are given by:

$$B_n = \frac{2}{L} \int_0^L \sin(\pi n x / L) \, f(x) \, \mathrm{d}x. \tag{11.55}$$

Hence, since $f(x)$ is piecewise continuous on $[0, 2]$ we can write

$$f(x) = \sum_{n=1}^{\infty} \left[ B_n \sin(\pi n x / L) \right]. \tag{11.56}$$

The RHS will converge when $x = 0$ and $x = 2$ to 0 (which is $f(0)$ or $f(2)$).

Hence in fact the series converges to $f(x)$ on the whole interval.

**Exercise:** Explicitly evaluate the $B_n$ and check these claims.

- The Fourier *cosine* series for the *same* function:

  The coefficients are given by:

$$A_{n\neq 0} = \frac{2}{L}\int_0^L \cos(\pi nx/L)\ f(x)\ \mathrm{d}x; \tag{11.57}$$

$$A_0 = \frac{1}{L}\int_0^L \hat{f}(x)\ \mathrm{d}x = 1/2. \tag{11.58}$$

  Hence, since $f(x)$ is piecewise continuous on $[0, 2]$ we can write

$$f(x) = \sum_{n=0}^{\infty} [A_n \cos(\pi nx/L)] \tag{11.59}$$

  The RHS will converge when $x = 0$ and $x = 2$ to 0 (which is $f(0)$ or $f(2)$).

  Hence in fact the series in this case converges to $f(x)$ on the whole interval.

  **Exercise:** Explicitly evaluate the $A_n$ and check these claims.

## 11.8 Exercises

### 11.8.1 Some Fourier work

Recall the Fourier theorems given out in the lecture topic notes.

In each of the cases below, find the indicated Fourier series for the given function, and, on the same diagram on which you have sketched the function, sketch the first four partial sums (and so watch the series gradually converge to the function).

a. $f(x) = x^2$ for $0 < x < 1$. Find a sine series.

b. $f(x) = x^2$ for $0 < x < 1$. Find a cosine series.

c. $f(x) = 1$ for $0 < x < 1$; $f(x) = -1$ for $-1 < x < 0$. Find the full Fourier series.

d. $g(x) = \sin x$ for $0 < x < \pi$. Find a Fourier cosine series.

e. $h(x) = \sin(3\pi x)$ for $0 < x < 1$. Find a Fourier sine series.

Naturally, Maple will be incredibly helpful for drawing the partial sums, and doing integrals!!

### 11.8.2 Fourier sine and cosine series

Consider the function $f(x) = \cos(2x)$ for $x \in (0, \pi)$.

1. Find a Fourier cosine series for $f(x)$.
2. Find a Fourier sine series for $f(x)$.

The remaining questions illustrate how you must use the cunning and brilliance honed over years of struggling through Maths courses to solve the problem. And your common sense.

### 11.8.3 Heat equation using Fourier series

**Boyce and DiPrima, Chapter 10.5, problem 5.**

This illustrates how to deal with the case where the end temperatures are kept fixed, but not at zero degrees. You should consult the relevant part of Boyce and DiPrima.

Let an aluminium rod of length l be initially at the uniform temperature of 25C. Suppose that at time $t = 0$ the end $x = 0$ is cooled to 0C while the end $x = L$ is heated to 60C, and that both ends are thereafter maintained at those temperatures.

a. Find the temperature distribution in the rod at any time $t$.
Now assume that $L = 20$ cm.

b. Use only the first term in the series for the temperature $U(x, t)$ to find the approximate temperature at $x = 5$ when $t = 30$ sec, and when $t = 60$ sec.

c. Use the first two terms for the series for $U(x,t)$ to find an approximate value of $U(5,30)$. What is the percentage difference between the one- and the two- term approximations? Does the third term in the series have any appreciable effect for this value of $t$?

d. Use the first term in the series for $U(x,t)$ to estimate the time that must elapse before the temperature at $x = 5$ comes within 1% of its steady state value.

### 11.8.4 Heat equation

**Boyce and DiPrima, chapter 10.5, problem 10.**

Another heat bar problem, this time with a mixture of end conditions. Find the steady state temperature in a bar that is insulated at the end $x = 0$ and held constant at the end $x = L$.

**Question 10** *What does this mean physically?*

### 11.8.5 Heat equation

Consider the heat equation

$$\partial_t U(t,x) = \partial_x^2 U(t,x), \tag{11.60}$$

subject to the boundary conditions

$$U(t,-L) = 0 = U(t,L); \tag{11.61}$$

$$U(0,x) = f(x) = f(-x). \tag{11.62}$$

That is, $f(x)$ is an even function of $x$.

1. Using separation of variables find a series representation for $U(t,x)$ that satisfies the boundary conditions at $\pm L$.

2. Then specify the value of the various coefficients in the series on terms of $f(x)$, the initial data at $t = 0$.

### 11.8.6 Laplace's equation

**Boyce and DiPrima, chapter 10.7, problem 6.**

This problem requires you to write Laplace's equation in terms of polar coordinates, and then solve by separation of variables.

Find the solution $u(r, \theta)$ of Laplace's equation in the circular sector $r < a$, $0 \le \theta \le \pi$, also satisfying the BC

$$u(r, 0) = 0; \tag{11.63}$$

$$u(r, \pi) = 0 \quad \text{for} \quad 0 < r < a; \tag{11.64}$$

$$u(a, \theta) = f(\theta) \quad \text{for} \quad 0 \le \theta \le \pi. \tag{11.65}$$

Assume that $u$ is single-valued and bounded in the given region.

In the problem, take

$$f(\theta) = \sin^2(2\theta). \tag{11.66}$$

Consider $u(r, \theta)$ to be the equilibrium temperature in the sector, when its radial sides are kept fixed at zero degrees, and the arc is heated according to $f(\theta)$.

### 11.8.7 Laplace's equation

Use Fourier series to solve Laplace's equation in the square $0 < x, y < 1$ satisfying the boundary conditions

$$U(0, y) = 0; \tag{11.67}$$

$$U(1, y) = 10; \tag{11.68}$$

$$U(x, 0) = 20; \tag{11.69}$$

$$U(x, 1) = 40x(1 - x) = f(x); \tag{11.70}$$

corresponding to the case of the equilibrium distribution of temperature in a square of gold with edges kept at temperatures of 0, 10, 20, and $f(x)$ degrees respectively.

You will find problems 3, 4 of chapter 10.7 of Boyce and DiPrima very useful, in that they indicate how to deal with the non-zero temperatures.

# Chapter 12

# Gibbs phenomenon

From the various Maple worksheets we have seen, it is clear that the "squiggles", the Gibbs phenomenon, have to do with discontinuities in the function $f(x)$...

But for the Fourier theorem to apply $f(x)$ must be piecewise continuous...

Therefore:

$$f(x) = \text{(continuous and periodic)} + \text{(finite number of finite discontinuities)}$$

With regards to the Gibbs phenomenon we need only focus on the:

$$\text{(finite number of finite discontinuities)}$$

But since the process of calculating the Fourier coefficients, and summing the Fourier series is linear, there is no loss of generality in focussing on just a single one of these discontinuities. In fact, there is really no loss of generality in considering a step discontinuity:

$$f(x) = \text{signum}(x-a) = \begin{cases} +1 & x > a \\ 0 & x = a \\ -1 & x < a \end{cases} .$$

For simplicity, (ie, good enough for most purposes), we set $a = 0$,
so that we consider

$$f(x) = \text{signum}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} .$$

We saw, in one of the Maple worksheets, that $a \neq 0$ was qualitatively similar to $a = 0$.

The function

$$f(x) = \text{signum}(x) = \begin{cases} +1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

is odd, so the natural thing to do is consider a sine series...

We might as well work on the unit interval $[-1, +1]$.

The Fourier coefficients are

$$A_n = 2\int_0^1 1 \cdot \sin(n\pi x)\mathrm{d}x = \frac{2}{n\pi}[-\cos(n\pi x)]_0^1 = \frac{2}{n\pi}\{1 - \cos(n\pi)\}$$

That is

$$A_{2m} = 0; \qquad A_{2m+1} = \frac{4}{\pi(2m+1)}$$

Therefore

$$\text{signum}(x) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin([2m+1]\pi x)}{2m+1}$$

This Fourier series converges to $+1$ for $x > 0$, converges to $0$ at $x = 0$, and converges to $-1$ for $x < 0$.

Now define the finite sum

$$S_M(x) = \frac{4}{\pi}\sum_{m=0}^{M} \frac{\sin([2m+1]\pi x)}{2m+1}$$

That is

$$S_M(x) = \frac{4}{\pi}\left\{\sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \ldots + \frac{\sin([2M+1]\pi x)}{2M+1}\right\}$$

For each fixed $x$ we have

$$\lim_{M\to\infty} S_M(x) = \text{signum}(x)$$

This is the content of the Fourier convergence theorem...

But what else can we say about this series

$$S_M(x) = \frac{4}{\pi}\left\{\sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \ldots + \frac{\sin([2M+1]\pi x)}{2M+1}\right\}$$

Let's evaluate this sum at the specific $M$-dependent point

$$x = \frac{1}{2[M+1]}.$$

Then:

$$S_M\left(\frac{1}{2[M+1]}\right) = \frac{4}{\pi}\sum_{m=0}^{M}\frac{\sin\left(\frac{[2m+1]\pi}{2[M+1]}\right)}{2m+1}$$

That is:

$$S_M\left(\frac{1}{2[M+1]}\right) = \frac{4}{\pi}\sum_{m=0}^{M}\frac{\sin\left(\frac{[2m+1]\pi}{2[M+1]}\right)}{\frac{2m+1}{2[M+1]}}\frac{1}{2[M+1]}$$

That is:

$$S_M\left(\frac{1}{2[M+1]}\right) = \frac{2}{\pi}\sum_{m=0}^{M}\frac{\sin\left(\frac{[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}}\frac{1}{M+1}$$

But note that the sum is just the mid-point Riemann sum for approximating the integral

$$\sum_{m=0}^{M}\frac{\sin\left(\frac{[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}}\frac{1}{M+1} \approx \int_0^1 \frac{\sin(\pi u)}{u}\mathrm{d}u$$

Note that $\sin(\pi u)/u$ is continuous... So it is certainly Riemann integrable... Therefore the limit $M \to \infty$ exists, and we have:

$$\lim_{M\to\infty} S_M\left(\frac{1}{2[M+1]}\right) = \frac{2}{\pi}\int_0^1 \frac{\sin(\pi u)}{u}\mathrm{d}u = \frac{2\ \mathrm{Si}(\pi)}{\pi}$$

(There are many other ways of getting to the same conclusion.) Numerically:

$$\lim_{M\to\infty} S_M\left(\frac{1}{2[M+1]}\right) = \frac{2\ \mathrm{Si}(\pi)}{\pi} = 1.178979744 > 1.$$

So there is guaranteed to be an overshoot... Since the gap from $-1$ to $+1$ is 2, the fractional overshoot is

$$\Delta = \frac{\frac{2\,\mathrm{Si}(\pi)}{\pi} - 1}{2} = \frac{\mathrm{Si}(\pi)}{\pi} - \frac{1}{2} = 0.0894898720 \approx 9\%$$

This 9% overshoot is the Gibbs phenomenon...
(Though it should really be called the Wilbraham phenomenon.)

Let us now consider a slightly more general idea:

$$S_M\left(\frac{w}{2[M+1]}\right) = \frac{4}{\pi}\sum_{m=0}^{M} \frac{\sin\left(\frac{w[2m+1]\pi}{2[M+1]}\right)}{2m+1}$$

Repeating the analysis (with trivial modifications) we see:

$$S_M\left(\frac{w}{2[M+1]}\right) = \frac{2}{\pi}\sum_{m=0}^{M} \frac{\sin\left(\frac{w[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}} \frac{1}{M+1}$$

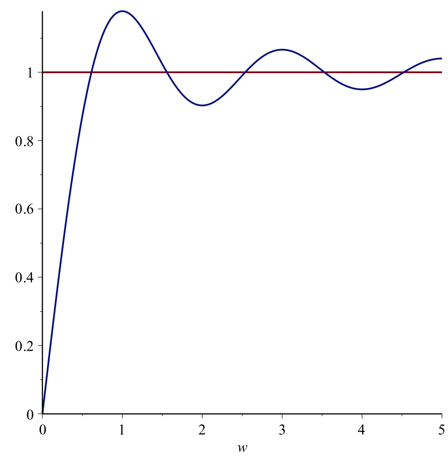
But note that the sum is just the mid-point Riemann sum for approximating the integral

$$\sum_{m=0}^{M} \frac{\sin\left(\frac{w[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}} \frac{1}{M+1} \approx \int_0^1 \frac{\sin(w\pi u)}{u}\mathrm{d}u$$

Therefore:

$$\lim_{M\to\infty} S_M\left(\frac{w}{2[M+1]}\right) = \frac{2}{\pi}\int_0^1 \frac{\sin(w\pi u)}{u}\mathrm{d}u = \frac{2\,\mathrm{Si}(w\pi)}{\pi}$$

(There are many other ways of getting to the same conclusion.)




$$\lim_{M\to\infty} S_M\left(\frac{w}{2[M+1]}\right) = \frac{2\ \mathrm{Si}(w\pi)}{\pi}$$

That is, near a discontinuity we have:

$$S_M(x) \approx \frac{2\ \mathrm{Si}(2\pi x[M+1])}{\pi}$$

This is actually a reasonably good approximation, at least as long as you are closer to the discontinuity at $x = 0$ than you are to the other discontinuity at $x = \pm 1$.

We can also argue as follows. Consider

$$S_M(x) = \frac{4}{\pi}\sum_{m=0}^{M} \frac{\sin([2m+1]\pi x)}{2m+1} = 4\int_0^x \sum_{m=0}^{M} \cos([2m+1]\pi u)du$$

Then performing the sum

$$S_M(x) = 2\int_0^x \frac{\sin([2M+2]\pi u)}{\sin(\pi u)}du$$

For $|x| \ll 1$ we have $|u| < |x| \ll 1$ so $\sin(\pi u) \approx \pi u$ and

$$S_M(x) \approx \frac{2}{\pi}\int_0^x \frac{\sin([2M+2]\pi u)}{u}du$$

Change variables:

$$S_M(x) \approx \frac{2}{\pi} \int_0^{[2M+2]\pi x} \frac{\sin u}{u} du$$

That is

$$S_M(x) \approx \frac{2}{\pi} \operatorname{Si}\left(2[M+1]\pi x\right)$$

as before...
This is the Gibbs phenomenon, generic to discontinuous functions.
Similar things happen for the sawtooth function.

# Chapter 13

# Eigenfunction expansions

Some brief comments to give you a flavour of what can be done.

## 13.1 Basics

I will not prove any of this but I will simply assert that Fourier series are a very special case of the sort of things that happen with linear ODEs (and linear PDEs that are separable).

Any time you have a linear ODE of the form

$$DU = \lambda\, U, \tag{13.1}$$

with suitable boundary conditions, the solutions of this eigenvalue problem

$$\{U_\alpha(x);\ \lambda_\alpha\} \tag{13.2}$$

form a complete basis for a large set of functions defined on the domain of the ODE. Generically sums like

$$U(x) = \sum_\alpha A_\alpha\, U_\alpha(x) \tag{13.3}$$

can be used to construct all interesting functions on the domain of the ODE.

In particular, working with the 2-dimensional Laplacian in polar coordinates leads (after separation of variables) to Bessel's differential equation, the solutions of which (naturally enough) are Bessel functions. But this then suggests that we should be able to write Bessel series of the form

$$f(x) = \sum_\alpha A_\alpha\, J_m(\lambda_\alpha x). \tag{13.4}$$

Here $m$, the index of the Bessel function, is related to the number of dimensions of space, while the eigenvalues $\alpha$ are determined by boundary conditions such as (for example)

$$J_m(\lambda_\alpha R) = 0. \tag{13.5}$$

Convergence and orthogonality properties for these Bessel series (sometimes called Fourier–Bessel series) can be proved by techniques analogous to this used for the ordinary Fourier series.

Similar games can then be played with the Laplacian in 3 dimensions, leading to spherical harmonics and spherical Bessel functions.

Ditto for the Schroedinger equation for the simple harmonic oscillator which leads to Hermite polynomials.

Eigenfunction expansions are ubiquitous.

They underlie much of "special function theory" as the special functions are typically defined in terms of the PDE/ ODE you are trying to solve.

# Appendix A

# Localized waves

I will discuss this topic only if there is still time.

Localized waves are classical solutions of the wave equation that are partially localized in space or time, this localization generally coming at a cost such as infinite total energy and/or instability (leading to dispersion or diffraction). The catalogue of known localized waves is large and growing, but most of the known examples are not in an easy to digest form.

In this chapter I will exhibit a particularly simple "physical wavelet". It satisfies the properties that:

- It is a localized wave that solves the wave equation.
- The field is everywhere finite and nonsingular, and has quadratic falloff in both space and time.

These physical wavelets can be constructed for both complex and real scalar fields. The simplest case is that of the complex scalar field.

## A.1 Complex scalar field

The field configuration is

$$\phi(x) = -\frac{\phi_0 \; a^2}{[t - ia]^2 - x^2 - y^2 - z^2}. \tag{A.1}$$

That is

$$\phi(x) = \frac{\phi_0 \; a^2}{r^2 - t^2 + a^2 + 2iat}. \tag{A.2}$$

It is a straightforward exercise to verify that the wave equation is satisfied. To see that the field is everywhere bounded note

$$\begin{aligned} |\phi|^2 &= \frac{|\phi_0|^2\; a^4}{(r^2-t^2+a^2)^2+4a^2t^2} = \frac{|\phi_0|^2\; a^4}{(r^2+t^2+a^2)^2-4r^2t^2} \\ &\leq \frac{|\phi_0|^2\; a^4}{(r^2+t^2+a^2)^2-(r^2+t^2)^2} = \frac{|\phi_0|^2\; a^4}{a^4+2a^2(r^2+t^2)} \leq |\phi_0|^2. \end{aligned} \tag{A.3}$$

From the penultimate inequality we also derive

$$|\phi|^2 \leq \frac{1}{2}\; |\phi_0|^2\; \frac{a^2}{r^2+t^2}, \tag{A.4}$$

demonstrating the promised quadratic falloff in both space and time. Indeed for fixed $t$ the magnitude of the field is maximized when

$$r^2 = \max\{t^2-a^2, 0\}, \tag{A.5}$$

showing that the configuration disperses to spatial infinity at both $t \to \pm\infty$. In summary, what we have is a singularity-free exact localized solution to the d'Alembertian equation.

One way of guessing that the field configuration above is worth investigating is the following: It is easy to convince oneself that in 4 Euclidean dimensions the solution to Laplace's equation with a delta function source at the origin is

$$\phi(x) \propto \frac{1}{x^2+y^2+z^2+t^2}. \tag{A.6}$$

Thus in (3+1) Lorentzian dimensions the [singular] solution to the wave equation is

$$\phi(x) \propto \frac{1}{x^2+y^2+z^2-t^2}. \tag{A.7}$$

If the center of the pulse is now moved to a complex position $(0,0,0,0) \to (-ia,0,0,0)$ we have

$$\phi(x) \propto \frac{1}{x^2+y^2+z^2-(t-ia)^2}. \tag{A.8}$$

which is still a singular field configuration. This style of approach has been particularly advocated by Lekner.

## A.2 Real scalar field

By taking real and imaginary parts of the complex solution above we can write down two solutions for the real scalar field. Namely

$$\phi_1 = \frac{\phi_0 \; a^2 \; \{t^2 - r^2 - a^2\}}{(t^2 - r^2 - a^2)^2 + 4a^2t^2}; \tag{A.9}$$

$$\phi_2 = \frac{\phi_0 \; a^2 \; 2at}{(t^2 - r^2 - a^2)^2 + 4a^2t^2}. \tag{A.10}$$

The physical wavelet discussed in this chapter is important because it represents a qualitatively different extended field configuration of a type not normally encountered in mathematical physics.

# Appendix B

# More examples of named PDEs

- Inviscid Burger's equation:

$$UU_x + U_y = 0. \tag{B.1}$$

- Telegraphers' equation:

$$u_{tt} = u_{xx} + \alpha\, u_x + \beta\, u. \tag{B.2}$$

- Clairaut's equation:

$$U = xU_x + yU_y + f(U_x, U_y). \tag{B.3}$$

- Minimal surface equation:

$$\partial_x \left[ \frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}} \right] + \partial_y \left[ \frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}} \right] = 0. \tag{B.4}$$

Study these equations and classify then as to order, linearity, quasi-linearity, whether or not they are (generalized) Euler equations, Euler type, *etcetera*.

Whenever possible, find general solutions.