Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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Outline:

1. Administrivia
2. The Euler equation
   - Basic definitions
   - Transformation of coordinates
   - Transformed Euler equation
     - Distinct roots
     - Coincident roots
     - Degenerate quadratic
   - Summary of general solution to the Euler equation
   - Euler type (elliptic/parabolic/hyperbolic)
   - Challenges
   - Exercises
3. Generalized constant-coefficient Euler PDE
4. Specific variable-coefficient extension of Euler's PDE
5. Tricomi's equation

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

The Euler equation

The Euler equation:

The Euler equation is a PDE that encompasses a wide variety of phenomena — that's why we are going to spend quite some time discussing both it and its general solutions.

Definition

The Euler PDE is

$$a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} = 0$$

where a, b, and h are (for the time being) constants.

[They could in general be taken as functions of x and y, but not just yet!].

The Euler equation:

We shall now rewrite this equation in a form for which the general solution will be obvious.

Warning

This is not the Euler equation of fluid mechanics.
That is a rather different beastie.
See previous chapter.

Comment

Note that this version of the Euler equation is a linear second-order PDE with constant coefficients.

Transformation of coordinates

Consider a linear transformation of the coordinates
(that is, the independent variables x and y)
to new independent variables s, t, defined as follows:

$$s = x + c\ y$$

$$t = x + d\ y$$

We shall now rewrite the Euler equation in terms of these new independent variables, and then cunningly choose the parameters c and d so that the resulting equation is really easy to solve.

The Euler equation:

- Note the Jacobian determinant is

$$\frac{\partial(s,t)}{\partial(x,y)} = \det \begin{bmatrix} \frac{\partial s}{\partial x} & \frac{\partial s}{\partial y} \\ \frac{\partial t}{\partial x} & \frac{\partial s}{\partial y} \end{bmatrix} = \det \begin{bmatrix} 1 & c \\ 1 & d \end{bmatrix} = d - c.$$

- The change of coordinates is proper (non-singular) as long as the determinant is non-zero.
- The change of coordinates is proper (non-singular) as long as $c \neq d$.

The Euler equation:

We have (by the multi-variable chain rule):

$$\frac{\partial U}{\partial x} = \frac{\partial U}{\partial s}\,\frac{\partial s}{\partial x} + \frac{\partial U}{\partial t}\,\frac{\partial t}{\partial x} = \frac{\partial U}{\partial s} + \frac{\partial U}{\partial t}$$

Equivalently,

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s} + \frac{\partial}{\partial t}$$

Similarly,

$$\frac{\partial}{\partial y} = c\frac{\partial}{\partial s} + d\frac{\partial}{\partial t}.$$

The Euler equation:

Hence

$$
\begin{aligned}
U_{xx} &= \left[\frac{\partial}{\partial x}\right]\left[\frac{\partial}{\partial x}\right] U \\
&= \left[\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right]\left[\frac{\partial}{\partial s}+\frac{\partial}{\partial t}\right] U \\
&= U_{ss} + 2U_{st} + U_{tt}
\end{aligned}
$$

Similarly

$$
U_{yy} = c^2\ U_{ss} + 2cd\ U_{st} + d^2\ U_{tt}
$$

$$
U_{xy} = c\ U_{ss} + (c+d)\ U_{st} + d\ U_{tt}
$$

The Euler equation:

Combining these results we easily see:

$$
\boxed{
\begin{aligned}
a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = & \\
& (a + 2hc + bc^2)\ U_{ss} + 2(a + h(c+d) + bcd)\ U_{st} \\
& + (a + 2hd + bd^2)\ U_{tt}.
\end{aligned}
}
$$

Leading to the transformed Euler equation (TEE):

$$
\boxed{(a + 2hc + bc^2)\ U_{ss} + 2(a + h(c+d) + bcd)\ U_{st} + (a + 2hd + bd^2)\ U_{tt} = 0.}
$$

Some very cunning choices (VCC):

To solve the TEE we will make some crafty choices for the parameters c and d occurring in the change of variables.

The choices we shall make will depend on the solutions to the quadratic equation

$$a + 2hz + bz^2 = 0.$$

We start by supposing that b is nonzero, so this quadratic always has two solutions.

Distinct roots:

If this equation has two distinct solutions, say z_1 and z_2, then choose the constants c and d to be these solutions:

$$c = z_1; \qquad\qquad d = z_2.$$

Then we plainly have:

- $c + d =$ the sum of the solutions $= -2h/b$.
- $cd =$ the product of solutions $= ab$.
- The discriminant $4(h^2 - ab) \neq 0$.

Distinct roots:

Note that since the roots are distinct, the transformation is proper, (both the original x and y, and the new s and t, are independent variables).

The Euler equation becomes

$$2\left[a + h\left(-\frac{2h}{b}\right) + b\,\frac{a}{b}\right] U_{st} = 0$$

or

$$2\,\frac{2ab - 2h^2}{b}\;U_{st} = 0$$

whence

$$\boxed{U_{st} = 0}$$

since by hypothesis $h^2 - ab$ is not zero.

Distinct roots:

- This transformed PDE is, of course, easy to solve.
- Its general solution is

$$\boxed{U(s, t) = F(s) + G(t)}$$

where F and G are arbitrary functions.

- Therefore, as functions of x and y:

$$\boxed{U(x, y) = F(x + cy) + G(x + dy)}$$

Coincident roots:

- In the coincident-root case, the discriminant $4(h^2 - ab) = 0$, and the quadratic has the single solution $z = -h/b$.
- So let us choose d to be the single root, $d = -h/b$.
- The last term in the transformed Euler equation (TEE) then vanishes, and the coefficient of the second term is:

$$a + h(c + d) + bcd = a + hc - \frac{h^2}{b} - \frac{bh}{b}c = \frac{ab - h^2}{b} = 0.$$

- Hence the TEE reduces to

$$(a + 2hc + bc^2)\; U_{ss} = 0$$

Coincident roots:

- If we choose c to be different from d, (which we must do to keep the transformation proper, and so keep the independent variables s and t truly independent of each other), we have

$$\boxed{U_{ss} = 0}$$

which has the obvious general solution

$$\boxed{U(s,t) = sF(t) + G(t)}$$

where F and G are arbitrary functions.

- The choice of the value of c is up to you here — it can be anything except d, the (unique) solution to the quadratic.

- Therefore, as functions of x and y:

$$\boxed{U(x,y) = (x+cy)\, F(x+dy) + G(x+dy); \qquad c \neq d.}$$

Degenerate quadratic:

- When $b = 0$, the work above does not apply, as we no longer have a genuine quadratic in z.
- However, you can easily adapt the theory outlined above for a transformation

$$s = c\,x + y,$$

$$t = d\,x + y,$$

leading to the quadratic

$$az^2 + 2hz + b^2 = 0.$$

- Then so long as a is nonzero, the results indicated above, with the role of x and y interchanged, apply.

Degenerate quadratic:

- If it happens that both a and b are zero, then you just have the simple equation $U_{xy} = 0$ to solve: and this is an easy thing to do.

- (In fact we have already done it.)

Summary 1:

- If b is nonzero:
  - If $h^2 - ab \neq 0$:

$$U(x, y) = F(x + cy) + G(x + dy)$$

where c and d are the distinct solutions to the quadratic equation

$$a + 2hz + bz^2 = 0.$$

  - If $h^2 - ab = 0$:

$$U(x, y) = (x + cy)\; F(x + dy) + G(x + dy)$$

where d is the single solution to

$$a + 2hz + bz^2 = 0$$

and c is any constant not equal to d.

Summary 2:

- If a is nonzero:
  - If $h^2 - ab \neq 0$:

    $$U(x,y) = F(cx+y) + G(dx+y)$$

    where c and d are the distinct solutions to the quadratic equation

    $$az^2 + 2hz + b = 0.$$

  - If $h^2 - ab = 0$:

    $$U(x,y) = (cx+y)\, F(dx+y) + G(dx+y)$$

    where d is the single solution to

    $$az^2 + 2hz + b = 0$$

    and c is any constant not equal to d.

Summary 3:

- If both $a = 0$ and $b = 0$:
  - The solution is
    $$U(x, y) = F(x) + G(y)$$
    where F and G are arbitrary.

Question

Do we have to do anything special if the roots of the quadratic are complex?

Euler type:

We define the "Euler type" of an Euler PDE by looking at the matrix formed by the coefficients of the second-derivative terms

$$E = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

The reason this matrix is interesting is because it can be used to re-write the Euler equation as

$$[\, \partial_x, \partial_y \,] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} \partial_x \\ \partial_y \end{bmatrix} U = a\, U_{xx} + 2h\, U_{xy} + b\, U_{yy} = 0$$

Now consider the determinant of this matrix and use it to classify Euler equations into the three classes:

- Elliptic: If the determinant $\det(E)$ is positive.
- Parabolic: If the determinant $\det(E)$ is zero.
- Hyperbolic: If the determinant $\det(E)$ is negative.

The reason for the terminology will be a bit mysterious at this stage.

Note that the determinant $\det(E) = ab - h^2$ is the negative of the discriminant occurring in the quadratic equation we used to simplify the Euler equation when finding the general solution.

Euler type:

Thus for Euler equations we can re-phrase the classification in terms of the algebraic equation:

$$[\,1,\ z\,] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} 1 \\ z \end{bmatrix} = a + 2h\, z + b\, z^2 = 0$$

Elliptic: If the roots are complex.

Parabolic: If the roots are coincident.

Hyperbolic: If the roots are real.

Euler type:

Once you go through the analysis leading to the general solution this leads to the characterization:

- Elliptic: If the general solution involves arbitrary functions of two distinct complex variables.
- Parabolic: If the general solution involves arbitrary functions of only one real variable.
- Hyperbolic: If the general solution involves arbitrary functions of two distinct real variables.

Euler type:

Warning

I should warn you that while the words Elliptic/ Parabolic/ Hyperbolic are most commonly used within the context of Euler's equation, (and its generalization with first order and linear terms as previously discussed), the notion is much more general.

Extending the Elliptic/ Parabolic/ Hyperbolic distinction to variable coefficients (so that the matrix $E(x, y)$ is position dependent) is easy.

Extending it to more dimensions is also easy.

Euler type:

Warning

It is less straightforward, but sometimes still possible and useful, to extend the Elliptic/ Parabolic/ Hyperbolic distinction to nonlinear PDEs and to systems of PDEs.

See, for instance, Courant and Hilbert for details.

Challenges:

For a challenge here's a few questions to think about —

Question (Terminology)

What is the origin of the terminology Elliptic/ Parabolic/ Hyperbolic?

Question (Terminology)

Are the terms Elliptic/ Parabolic/ Hyperbolic exclusive?

Question (Terminology)

Are the terms Elliptic/ Parabolic/ Hyperbolic complete?
(Do they cover all the possibilities?)

Challenges:

Question (Eikonal)

What is the meaning of the word "eikonal"?

Question (Symbol)

What is the "symbol" of a PDE?

Question (Fresnel equation)

What is the "Fresnel equation" of a PDE?

Exercises — Euler type:

Determine the Euler type (i.e., elliptic, hyperbolic, or parabolic) of each of the following PDEs, and obtain the general solution in each case:

a. $3U_{xx} + 4U_{xy} - U_{yy} = 0.$
b. $U_{xx} - 2U_{xy} + U_{yy} = 0.$
c. $4U_{xx} + U_{yy} = 0.$
d. $U_{xx} + 4U_{xy} + 4U_{yy} = 0.$
e. $U_{yy} + 2U_{xx} = 0.$
f. $4U_{xx} + U_{yy} = 0.$
g. $4U_{,xx} - U_{,yy} = 0.$
h. $4U_{,xx} + U_{,xy} + U_{,yy} = 0.$
i. $9U_{,xx} + 3U_{,xy} + U_{,yy} = 0.$
j. $8U_{,xx} + 3U_{,xy} + U_{,yy} = 0.$
k. $4U_{,xx} + 2U_{,xy} + U_{,yy} = 0.$

Generalized constant-coefficient Euler PDE:

Definition

One simple way of generalizing the Euler PDE is this:

$$a\,U_{xx} + 2h\,U_{xy} + b\,U_{yy} + c\,U_x + d\,U_y + e\,U + f = 0$$

where a, b, h, and c, d, e, f are constants, (and at least one of the second-order coefficients a, b, or h, is nonzero).

Generalized constant-coefficient Euler PDE:

Comment

This is still a linear second-order PDE with constant coefficients.

This generalization is not really as painful as it looks.

If the coefficients are constants the general solution can sometimes be found using modifications of the preceding argument.

Even then, sometimes there is no closed-form general solution, even for this constant coefficient case.

Project (Generalized constant-coefficient Euler PDE:)

- *Analyze this generalized constant-coefficient Euler PDE in detail.*
- *Completely classify those situations for which closed-form general solutions (in terms of two arbitrary functions) can be written down.*
- *Even when completely general solutions cannot be explicitly written down, it is often possible to find reasonably general classes of specific solution.*
- *Do as much as possible...*

Specific variable-coefficient extension of Euler's equation:

As an example, show that

$$u(x, y) = f(2x + y^2) + g(2x - y^2)$$

is a general solution to the equation

$$y^2 u_{xx} + \frac{1}{y} u_y - u_{yy} = 0$$

where f and g are arbitrary differentiable functions.

This is a specific example of a variable-coefficient extension of the Euler equation.

Is it elliptic, parabolic, or hyperbolic?

Later on, we shall have a lot more to say about this class of PDEs.

Tricomi's equation:

Consider Tricomi's PDE:

$$y\ U_{xx} + U_{yy} = 0.$$

Is it elliptic, parabolic, or hyperbolic?

Try to find a general solution to this PDE...

(Don't be surprised to find it's impossible, at least at this stage of the course.
By the end of the course you will see techniques powerful enough to write down a general solution for this PDE.)

We will have a lot more to say about this class of PDEs later.

End: