Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Euler equation with variable coefficients

Euler equation with variable coefficients:

It is often useful to consider a further extension of the definition of the Euler PDE:

Definition

The generalized variable-coefficient Euler PDE is

$$a(x,y)\,U_{xx} + 2h(x,y)\,U_{xy} + b(x,y)\,U_{yy} + c(x,y)\,U_x + d(x,y)\,U_y$$
$$+e(x,y)\,U + f(x,y) = 0,$$

where a, b, h, and c, d, e, f are functions of x and y.

(And at least one of the second-order coefficients, (a, b, or h), is not identically zero.)

(And we are explicitly staying in 2 dimensions.)

- This is not really as painful as it looks.
- Note that this is simply another name for the most general linear second-order PDE.
- First let's simultaneously focus attention on the second-order derivatives, and generalize the Euler equation even further by allowing for a nonlinear source term.
- Consider the form below.

Euler equation with variable coefficients:

Definition

The generalized variable-coefficient Euler PDE (with non-linear source) is

$$a(x,y)\; U_{xx} + 2h(x,y)\; U_{xy} + b(x,y)\; U_{yy} = F(x,y,U,U_x,U_y),$$

where a, b, are functions of x and y, and F is a function of its indicated arguments.

(And at least one of the second-order coefficients, (a, b, or h), is not identically zero.)

(And we are explicitly staying in 2 dimensions.)

- This is still less general than the class of quasi-linear Euler PDEs, see below.

Canonical form:

A remarkable result, in 2-dimensions, is that by a change of coordinates the variable coefficients of the second-order terms can always be made constant, and the Euler equation can always be brought into a simple canonical form.

Canonical form:

Theorem

In 2 dimensions, as long as $a(x,y)$, $h(x,y)$, and $b(x,y)$ are not all zero, you can always divide the plane into disjoint regions in each of which you can, by change of independent variables, bring the generalized variable-coefficient Euler PDE

$$a(x,y)\; U_{xx} + 2h(x,y)\; U_{xy} + b(x,y)\; U_{yy} = F(x,y,U,U_x,U_y),$$

into the form

$$U_{\bar{x}\bar{x}} + \epsilon\; U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x},\bar{y},U,U_{\bar{x}},U_{\bar{y}}),$$

where $\epsilon = \pm 1$ or 0, and $\tilde{F}$ is a function of its indicated arguments.

Furthermore

$$\epsilon = sign\left[a(x,y)\; b(x,y) - h(x,y)^2\right].$$

Canonical form:

- This theorem generalizes what we were already able to do with the constant-coefficient case.
- The existence of this theorem is one of the reasons the 2-dimensional Laplace and wave equations are of such fundamental importance.
- Note that

$$\det \begin{bmatrix} a(x,y) & h(x,y) \\ h(x,y) & b(x,y) \end{bmatrix} = a(x,y)\; b(x,y) - h(x,y)^2$$

can still be used to classify the PDE as elliptic, parabolic, or hyperbolic, but that this is now a position-dependent classification — the Euler type of the PDE can change from one part of the plane to another.

Canonical form:

Proof of the canonical form theorem:

Consider a change of variables from x, y to $\bar{x}$, $\bar{y}$.
Let

$$\bar{x} = \phi(x, y); \qquad \bar{y} = \psi(x, y).$$

Assume the change of variables is invertible (at least locally) so that

$$x = \Phi(\bar{x}, \bar{y}); \qquad y = \Psi(\bar{x}, \bar{y}).$$

By the inverse function theorem this will be true as long as the Jacobian is nonzero.

Canonical form:

That is, as long as

$$\frac{\partial(\bar{x},\bar{y})}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix} = \phi_x\psi_y - \phi_y\psi_x \neq 0.$$

Then

$$U(x,y) = U(\Phi(\bar{x},\bar{y}),\Psi(\bar{x},\bar{y})) = \bar{U}(\bar{x},\bar{y}).$$

Applying the (multi-variable) chain rule:

$$U_x = \bar{U}_{\bar{x}}\ \phi_x + \bar{U}_{\bar{y}}\ \psi_x;$$

$$U_y = \bar{U}_{\bar{x}}\ \phi_y + \bar{U}_{\bar{y}}\ \psi_y.$$

Canonical form:

Differentiating a second time,
(and applying the [multi-variable] chain rule a second time),
we see:

$$U_{xx} = \bar{U}_{\bar{x}\bar{x}}\ \phi_x^2 + 2\bar{U}_{\bar{x}\bar{y}}\ \phi_x\psi_x + \bar{U}_{\bar{y}\bar{y}}\ \psi_x^2 + \bar{U}_{\bar{x}}\ \phi_{xx} + \bar{U}_{\bar{y}}\ \psi_{xx};$$

$$U_{xy} = \bar{U}_{\bar{x}\bar{x}}\ \phi_x\phi_y + \bar{U}_{\bar{x}\bar{y}}\ (\phi_x\ \psi_y + \psi_x\ \phi_y) + \bar{U}_{\bar{y}\bar{y}}\ \psi_x\psi_y + \bar{U}_{\bar{x}}\ \phi_{xy} + \bar{U}_{\bar{y}}\ \psi_{xy};$$

$$U_{yy} = \bar{U}_{\bar{x}\bar{x}}\ \phi_y^2 + 2\bar{U}_{\bar{x}\bar{y}}\ \phi_y\psi_y + \bar{U}_{\bar{y}\bar{y}}\ \psi_y^2 + \bar{U}_{\bar{x}}\ \phi_{yy} + \bar{U}_{\bar{y}}\ \psi_{yy}.$$

Canonical form:

Now add and collect terms to obtain

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = \bar{a}\ \bar{U}_{\bar{x}\bar{x}} + 2\bar{h}\ \bar{U}_{\bar{x}\bar{y}} + \bar{b}\ \bar{U}_{\bar{y}\bar{y}} + \bar{e}\ \bar{U}_{\bar{x}} + \bar{f}\ \bar{U}_{\bar{y}},$$

where now

$$\bar{a} = a\,\phi_x^2 + 2h\,\phi_x\phi_y + b\,\phi_y^2;$$

$$\bar{h} = a\,\phi_x\,\psi_x + h\,(\phi_x\psi_y + \psi_x\phi_y) + b\,\phi_y\psi_y;$$

$$\bar{b} = a\,\psi_x^2 + 2h\,\psi_x\psi_y + b\,\psi_y^2;$$

$$\bar{e} = a\,\phi_{xx} + 2h\,\phi_{xy} + c\,\phi_{yy};$$

$$\bar{f} = a\,\psi_{xx} + 2h\,\psi_{xy} + c\,\psi_{yy}.$$

Canonical form:

This turns the original PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} = F(x, y, U, U_x, U_y),$$

into the form

$$\bar{a}\ \bar{U}_{\bar{x}\bar{x}} + 2\bar{h}\ \bar{U}_{\bar{x}\bar{y}} + \bar{b}\ \bar{U}_{\bar{y}\bar{y}} = F_2(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}),$$

But we now have the freedom to choose ϕ and ψ to make the transformed coefficients $\bar{a}$, $\bar{h}$, and $\bar{c}$, as simple as possible.

Canonical form:

- Start by choosing ϕ_x and ϕ_y so that $a\,\phi_x + h\,\phi_y \neq 0$; this can always be done.
- Then choose $\psi_y \neq 0$, and solve for $\bar{h} = 0$.
- Check that $\bar{a} \neq 0$.
- Now $\bar{h} = 0$ requires

$$\psi_x = -\psi_y\ \frac{h\phi_x + b\phi_y}{a\phi_x + h\phi_y}.$$

- We can check that these choices make sense by computing the Jacobian

$$\frac{\partial(\bar{x},\bar{y})}{\partial(x,y)} = \phi_x\psi_y - \phi_y\psi_x = \frac{\psi_y}{a\phi_x + h\phi_y}\ (a\,\phi_x^2 + 2h\,\phi_x\phi_y + b\,\phi_y^2)$$

$$= \frac{\psi_y}{a\phi_x + h\phi_y}\ \bar{a},$$

which is nonzero by hypothesis. (Thence implying $\bar{a} \neq 0$.)

Canonical form:

- But then $\bar{b}$ is easily computed to be

$$\bar{b} = \frac{\psi_y^2}{(a\phi_x + h\phi_y)^2}\,(ab - h^2)\,\bar{a}.$$

- So at this stage we have $\bar{h} = 0$ and we certainly know

$$\text{sign}(\bar{b}) = \text{sign}(ab - h^2)\,\text{sign}(\bar{a}).$$

- But the only thing we have used (so far) about ψ_y is that it is nonzero, so (provided $ab - h^2 \neq 0$) we are still free to pick

$$\psi_y = \frac{a\phi_x + h\phi_y}{\sqrt{|ab - h^2|}}.$$

- But then we have both $\bar{h} = 0$ and

$$\bar{b} = \text{sign}(ab - h^2)\ \bar{a}.$$

- But this final result works even if $ab - h^2 = 0$.
- So in this particular coordinate system the PDE is

$$\bar{a}(\bar{x}, \bar{y}) \left\{ \bar{U}_{\bar{x}\bar{x}} + \text{sign}(ab - h^2)\ \bar{U}_{\bar{y}\bar{y}} \right\} = F_2(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}).$$

- Dividing through by $\bar{a}$ now yields

$$\bar{U}_{\bar{x}\bar{x}} + \text{sign}(ab - h^2)\ \bar{U}_{\bar{y}\bar{y}} = F_3(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}).$$

Canonical form:

Now adopt the notation

$$\epsilon = \text{sign}(ab - h^2),$$

then

$$U_{\bar{x}\bar{x}} + \epsilon\, U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x}, \bar{y}, U, U_{\bar{x}}, U_{\bar{y}}),$$

and we are done.

QED!

Canonical form:

- Note that this works in any two dimensional region where $(ab - h^2)$ is of constant sign.
- This includes two dimensional regions where $(ab - h^2)$ is identically zero.
- Note that this is a “straightforward” extension of what we did for the constant-coefficient Euler equation.

Canonical form:

- If you want to consider a two dimensional region where $(ab - h^2)$ changes sign, the trick is to use $(ab - h^2)$ as one of your new coordinates, say $\bar{x}$.
- You can still eliminate $\bar{h}$ in the same way, but now

$$\bar{b} = \frac{\psi_y^2}{(a\phi_x + h\phi_y)^2} \, (ab - h^2) \, \bar{a} \to \frac{\psi_y^2}{(a\phi_x + h\phi_y)^2} \, \bar{x} \, \bar{a}.$$

- The further choice

$$\psi_y = a\phi_x + h\phi_y$$

now leads to

$$\bar{a}(\bar{x}, \bar{y}) \left\{ \bar{U}_{\bar{x}\bar{x}} + \bar{x} \, \bar{U}_{\bar{y}\bar{y}} \right\} = F_2(\bar{x}, \bar{y}, \bar{U}, \bar{U}_{\bar{x}}, \bar{U}_{\bar{y}}).$$

- We now rewrite this as

$$U_{\bar{x}\bar{x}} + \bar{x} \, U_{\bar{y}\bar{y}} = \tilde{F}(\bar{x}, \bar{y}, U, U_{\bar{x}}, U_{\bar{y}}),$$

which is Tricomi's equation with a nonlinear source.

Canonical form:

- Note what we have done — in two dimensions the second-derivative part of the general variable-coefficient Euler equation has been reduced to a very small number of standard cases:
  - — Wave equation (with nonlinear source),
  - — Laplace's equation (with nonlinear source),
  - — Parabolic equation (with nonlinear source),
  - — Tricomi's equation (with nonlinear source).
- This is a tremendous simplification.

Canonical form:

- This whole discussion can be given a “geometrical” interpretation which will not make any sense until know some differential geometry:
  - Any two-dimensional manifold with a non-singular metric tensor is locally conformally flat.
  - Any two-dimensional manifold with a Euclidean metric tensor is locally conformal to two-dimensional Euclidean space.
  - Any two-dimensional manifold with a Lorentzian metric tensor is locally conformal to two-dimensional Minkowski space.

Canonical form:

- Unfortunately if you go beyond 2 dimensions things get a whole lot more complicated.
  - In 3 dimensions you can at least diagonalize the matrix of coefficients of the second-order terms, but you cannot make the coefficients piecewise constant.
  - (Darboux's theorem for 3-manifolds — proving this is not easy.)
  - See for instance: http://www.u-gakugei.ac.jp/~sekizawa/diagonal.pdf.
  - Note the appeal to the Cauchy–Kowalevski Theorem...
  - The situation in 4 dimensions is even worse...
- The elliptic/ parabolic/ hyperbolic distinction requires more information than just the determinant of the matrix of second order coefficients — you now need to know the signature of that matrix, the number of positive, negative, and zero eigenvalues.

Canonical form:

- Elliptic/ parabolic/ hyperbolic:
  - If all the eigenvalues of the matrix of second-order coefficients are nonzero and have the same sign, then the PDE is elliptic.
  - If all the eigenvalues of the matrix of second-order coefficients are nonzero and some have differing sign, then the PDE is hyperbolic.
  - If all the eigenvalues of the matrix of second-order coefficients are nonzero and exactly one has a different sign from all the others, then the PDE is strictly hyperbolic.
  - If all the eigenvalues of the matrix of second-order coefficients are nonzero and at least two are positive while at least two are negative, (which can only happen in four or more dimensions), then the PDE is ultra-hyperbolic.
    (This is bad; effectively it means you have two time directions; very not good...)
  - If some of the eigenvalues of the matrix of second-order coefficients are zero, then the PDE is parabolic.

Examples — variable coefficient Euler:

Here are some examples of standard PDEs of considerable importance that fall under the heading of variable-coefficient Euler type.

Examples — variable coefficient Euler:

- Poisson:
$$\nabla^2\phi = \rho$$
Laplace's equation with a position-dependent source.
  - Electrostatic potential in the presence of electric charge.
  - Gravitational potential in the presence of matter.
  - Equilibrium temperature in the presence of heat sources.
  - Now also defined for curved space.
- In terms of the generalized Euler PDE
$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} + c\ U_x + d\ U_y + e\ U + f = 0$$
the Poisson equation corresponds to
$$a \to 1; \quad h \to 0; \quad b \to 1;$$
$$c \to 0; \quad d \to 0; \quad e \to 0; \quad f \to \rho(x,y).$$
- There is a natural generalization to three space dimensions.

Examples — variable coefficient Euler:

- Maxwell (now with sources):
  Adding charges and currents to the Maxwell equations

$$
\begin{aligned}
&\mathit{div}\, E = \rho \\
&\mathit{curl}\, B - \partial_t\, E = j \\
&\mathit{div}\, B = 0 \\
&\mathit{curl}\, E + \partial_t\, B = 0
\end{aligned}
$$

- In the presence of sources (and/ or curved space-time) the Maxwell equations can be put into the form of a system of generalized variable-coefficient Euler PDEs, with electric fields coupled to magnetic fields, charges, and currents.

Examples — variable coefficient Euler:

- Maxwell (now with sources):
  You can use the rules of vector calculus to derive wave equations for E and B:
  $$\partial_t^2 E - \nabla^2 E = grad\,\rho - \partial_t\, j$$
  $$\partial_t^2 B - \nabla^2 B = -curl\, j$$
- Note that for simplicity I have again adopted units where the speed of light equals unity, and that we are now dealing with wave equations with sources.

Examples — variable coefficient Euler:

- Schroedinger equation:

$$-i\hbar\frac{\partial}{\partial t}\psi(t,\vec{x}) = \left\{-\frac{\hbar^2}{2m}\nabla^2 + V(t,\vec{x})\right\}\psi(t,\vec{x})$$

- This PDE links the space and time dependence of the probability amplitude for finding a particle at a particular point.
- (Thankfully a linear PDE, which is why we can do such a lot with it.)
- This equation is very well understood and underlies much of humanity's quantum technology.

Examples — variable coefficient Euler:

- In terms of the generalized Euler PDE

$$a\ U_{xx} + 2h\ U_{xy} + b\ U_{yy} + c\ U_x + d\ U_y + e\ U + f = 0$$

the Schroedinger equation corresponds to

$$a \to 0; \quad h \to 0; \quad b \to +\frac{\hbar^2}{2m};$$

$$c \to -i\hbar; \quad d \to 0; \quad e \to -V(t,x); \quad f \to 0$$

with the notational change $x \to t$, $y \to x$.

- There is a natural generalization from (1+1) to (2+1) and (3+1) dimensions.

Examples — variable coefficient Euler:

- Continuity equation:
$$\partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$
Recall that this is a quasi-linear first order PDE.
Because there are no second-order derivatives, the continuity equation cannot be put into Euler form.

Examples — variable coefficient Euler:

- Euler (hydrodynamics):

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho}$$

Recall that this is a quasi-linear first order PDE.
Because there are no second-order derivatives, the hydrodynamic Euler equation cannot be put into Euler form.

Quasi-linear Euler PDE:

Definition

The generalized quasi-linear Euler PDE is

$$a(x, y, U, U_x, U_y)\; U_{xx} + 2h(x, y, U, U_x, U_y)\; U_{xy} + b(x, y, U, U_x, U_y)\; U_{yy}$$
$$= F(x, y, U, U_x, U_y),$$

where a, h, and b, are functions of x, y, U and its first derivatives, and F is a function of its indicated arguments.

(And at least one of the second-order coefficients a, b, or h, is not identically zero.)

Quasi-linear Euler PDE:

- Note that the quasi-linear Euler equation is simply another name for the general quasi-linear second order PDE.
- Note that if you classify the quasi-linear Euler equations into elliptic, parabolic, hyperbolic by looking at the sign of $ab - h^2$, then the Euler type can depend not only on where you are in space, but also on the value of the dependent variable and its derivatives at that point.

Quasi-linear Euler PDE:

Here are some examples of standard PDEs of considerable importance that fall under the heading of quasi-linear Euler type.

(Though the interpretation might sometimes be considered a bit strained.)

Quasi-linear Euler PDE:

- Navier–Stokes equation:

$$\partial_t \vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \frac{\vec{B}}{\rho} + \nu \nabla^2 \vec{v}$$

$$\vec{\nabla} \cdot \vec{v} = 0$$

  - This is Euler's fluid dynamic equation (Newton's second law), plus incompressibility, plus conservation of mass, plus a particular model for viscosity.
  - Because of the viscosity term there is now at least one second-order term in the PDE — and because this second-order derivative occurs linearly the first of the two PDEs can be viewed as a quasi-linear Euler PDE.
  - Indeed this is a parabolic PDE.

Quasi-linear Euler PDE:

- These equations look innocent; they are very difficult to analyze.
- The fact that they are nonlinear in the velocity field $\vec{v}$ is the ultimate source of all the difficulty.
- Remember I told you that EUS is extremely difficult to prove for generic PDEs?

Quasi-linear Euler PDE:

- There is currently a US$1,000,000 prize from the Clay Mathematics institute for "substantial progress towards proving existence and smoothness" of the solutions:
  Waves follow our boat as we meander across the lake, and turbulent air currents follow our flight in a modern jet. Mathematicians and physicists believe that an explanation for and the prediction of both the breeze and the turbulence can be found through an understanding of solutions to the Navier–Stokes equations. Although these equations were written down in the 19th Century, our understanding of them remains minimal. The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.

Quasi-linear Euler PDE:

- For the details of the challenge, see:
  http://www.claymath.org/millennium-problems
- Please do not present me with any prize claims;
  see the rules as given on the website.

Exercises on Euler PDEs:

Classify the following PDEs according to whether or not they are:

- Euler (simple, constant coefficient).
- Euler (generalized, constant coefficient).
- Euler (variable coefficient, possibly with nonlinear source).
- Euler (quasi-linear).
- Non-Euler.

Whenever they fall into one of the many Euler classes above, further classify them according to whether they are elliptic, parabolic, hyperbolic.

(For some of these PDEs it will simply be a matter of reading the notes and copying the answers I've already given.)

Exercises on Euler PDEs:

a. $V^2\ V_{xy} + V_x\ V_y + (x^2 - y^2)V = 3xy$.
b. $U_{xxz} - 2(x+z)U_{xyz} - U_{xx} + \sin(xyz)U_{xx} = \cos(U)$
c. $U_t - UU_{xx} + 12xU_x = U$.
d. $Y_{xxx} - \cos Y = Y_t$.
e. $V_{xt} - \sin V = \exp(x+t)$.
f. $Y_{xx} + \cos(xy)Y_{yxy} = Y + \ln(x^2 + y^3)$.
g. $U_t = U_{xx} - 12U\ U_x$.
h. $V_{yx} + V_x + V_y = V_{xyy}$.
i. $U_{tt} - \cos(U_x) = U$.
j. $\cos x \cdot U_x + \sin t \cdot U_t = U$.

Exercises on Euler PDEs:

k. Schrodinger equation (with potential):

$$-i\partial_t \psi = \frac{1}{2m}\nabla^2\psi + V(x)\psi.$$

l. Monge–Ampere equation (2 variable):

$$u_{xx}u_{yy} - u_{xy}^2 = f(x, y, u, u_x, u_y).$$

m. Monge–Ampere equation (multi-variable):

$$\det\left[\frac{\partial^2 u}{\partial x^i\, \partial x^j}\right] = f\left(x^i, u, \frac{\partial u}{\partial x^i}\right).$$

n. Navier–Stokes equation:

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{\vec{\nabla} p}{\rho} + \nu\nabla^2\vec{v}.$$

Exercises on Euler PDEs:

o. Tricomi equation:

$$y\ U_{xx} + U_{yy} = 0.$$

p. Frobenius–Mayer equation (special case, one dependent variable):

$$\frac{\partial U}{\partial x^i} = F_i(x, U).$$

q. Biharmonic equation:

$$\nabla^4 \Psi = 0.$$

That is, $(\nabla^2)^2\ \Psi = 0$, or more explicitly:

$$\left[\partial_x^2 + \partial_y^2 + \partial_z^2\right]^2 \Psi = 0.$$

Exercises on Euler PDEs:

r. Benjamin–Bona–Mahony equation:

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

s. Chaplygin equation:

$$u_{xx} + \frac{c^2\ y^2}{c^2 - y^2}\ u_{yy} + y\ u_y = 0.$$

t. Boussinesq equation:

$$u_{tt} - \alpha^2 u_{xx} = \beta^2 u_{xxtt}.$$

u. Euler–Darboux equation:

$$u_{xy} + \frac{\alpha\ u_x - \beta\ u_y}{x - y} = 0.$$

Exercises on Euler PDEs:

v. Korteweg–deVries–Burger:

$$u_t + 2uu_x - \nu \; u_{xx} + \mu \; u_{xxx} = 0.$$

w. Kirchever–Novikov equation:

$$\frac{u_t}{u_x} = \frac{1}{4}\frac{u_{xxx}}{u_x} - \frac{3}{8}\frac{u_{xx}^2}{u_x^2} + \frac{3}{8}\frac{4u^3 - g_2 u - g_3}{u_x^2}.$$

(Start by simplifying this a little.)

x. Lin–Tsien equation:

$$2u_{tx} + u_x \; u_{xx} - u_{yy} = 0.$$

y. Monge–Ampere equation (generalized):

$$E(x,y,U,U_x,U_y)\left[U_{xx}U_{yy}-U_{xy}^2\right]$$
$$+A(x,y,U,U_x,U_y)\;U_{xx}+B(x,y,U,U_x,U_y)\;U_{xy}+C(x,y,U,U_x,U_y)\;U_{yy}$$
$$+D(x,y,U,U_x,U_y)=0$$

or even more generally (multi variable case):

$$E(x^i,U,\partial_i U)\,\det\left[\frac{\partial^2 u}{\partial x^i\;\partial x^j}\right]+\sum_{ij}A^{ij}(x^i,U,\partial_i U)\;U_{,ij}+D(x^i,U,\partial_i U)=0.$$

z. Cauchy–Riemann system of PDEs:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y};$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

After answering the question for the Cauchy–Riemann system itself, iterate these Cauchy–Riemann equations to find a pair of PDEs that decouple — they depend only on u, and only on v, but not both.

End: