Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

Matt Visser

21 February 2024

Outline:

1. Administrivia
2. Separation of Variables: Pedagogical example
3. Separation of Variables: Comments
4. Separation of Variables: Sufficient conditions
5. Separation of Variables: Abstract examples
6. Separation of Variables: Exercises

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Separation of variables: Pedagogical example

Separation of Variables: Pedagogical example

The method of Separation of Variables (SOV) is a general technique of fundamental importance for solving PDEs

— I'll introduce it by looking at a specific example.

Example used for illustration:

The wave equation

$$U_{xx} - U_{tt} = 0$$

with conditions

$$\begin{aligned} U(x,0) &= f(x) && \text{(the initial shape of the string)} \\ U_t(x,0) &= g(x) && \text{(the initial velocity of the string)} \\ U(0,t) &= 0 && \text{(pinned endpoint)} \\ U(L,t) &= 0 && \text{(pinned endpoint)} \end{aligned}$$

This is a linear PDE.

The last two conditions are called homogeneous because they involve the dependent variable U and its derivative linearly and homogeneously.

The first two conditions are inhomogeneous.

The method:

1. Use a trial solution of the variable-separated form:

$$U(x,t) = X(x)\ T(t)$$

In the case of the example we have chosen we find

$$U_{xx} = X''(x)\ T(t)$$
$$U_{tt} = X(x)\ T''(t)$$

and so

$$X''(x)\ T(t) - X(x)\ T''(t) = 0$$

where $'$ stands for a derivative of the function with respect to its argument — either x or t as appropriate.

The method:

2. Separate the variables:

With luck, the PDE will allow you to gather all terms involving one independent variable on the left (x, say) and all other independent variables (t in this case) on the right hand side.

We have

$$X''(x)\ T(t) = X(x)\ T''(t).$$

Dividing both sides by $X(x)\ T(t)$, we find

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)}$$

The method:

Warning

If it turns out that you cannot separate the variables this way, then you will have to solve the DE some other way, because SOV won't work.

Warning

Though rare, sometimes you might have to try additive SOV

$$U(x,t) = X(x) + T(t).$$

The method:

3. Find corresponding ODEs:

Each of the separated terms must be a constant.
Thus you find two ODEs to solve, involving as yet arbitrary constants.

Use the fact that the only way you can have a function $F(x)$,
of one variable x, equal to another function $G(t)$,
of another independent variable t, is to have both functions constant.

Using this find ODEs that the separated functions must satisfy.

We have therefore:

$$\frac{X''(x)}{X(x)} = k \qquad \text{and} \qquad \frac{T''(t)}{T(t)} = k$$

where k is some constant (as yet to be determined).

The method:

Thus we obtain a pair of ODEs for the unknown functions X and T.

$$X'' = k\,X$$

$$T'' = k\;T$$

If there are more than two independent variables, you will need to continue the separation of variables procedure.
At the end, you should finish up with a collection of ODEs, one for each of the assumed functions in the separated variable form.

The method:

4. The ODEs can be solved in the usual way.

Doing so will give the functions $X(x)$ and $T(t)$ in terms of a selection of arbitrary constants:

$$X = A\, e^{\sqrt{k}x} + B\, e^{-\sqrt{k}x}$$

$$T(t) = C\, e^{\sqrt{k}t} + D\, e^{-\sqrt{k}t}$$

The method:

5. Apply homogeneous boundary conditions.

Now apply any homogeneous boundary conditions that you may have, in order to find out some information about the constants A, B, C, D, and k that are thinking about.

Note that if $U(x,t) = X(x)\ T(t)$, and if a condition is $U(0,t) = 0$, then we must have $X(0) = 0$, since we certainly do not want $T(t) = 0$.

(That would make $U(x,t)$ identically zero, which is uninteresting.)

The method:

More specifically:

If $T(t) = 0$, then $U(x, t) = 0$ for all values of t and x,
so we have the trivial solution,
[which obviously will not satisfy the remaining conditions,
which have $U(x, t)$ nonzero for some values of x and t].

Thus, we will always look for "non-trivial" solutions only.

At this point it becomes apparent that some values of k are acceptable, and others not.

The method:

Indeed, in the case we are treating here, k cannot in fact be positive:

- If $k > 0$, so we can write $k = b^2$ for some real b, then

$$X(x) = A\ e^{bx} + B\ e^{-bx}.$$

- Then the boundary conditions imply

$$\begin{aligned} X(0) = 0 \quad &: \quad A + B = 0 \\ X(L) = 0 \quad &: \quad A\ e^{bL} + B\ e^{-bL} = 0 \end{aligned}$$

  which has the unique solution $A = 0 = B$.

- But this solution would imply that $X(x) = 0$ for all x, and hence $U(x, t) = 0$ for all x and t — i.e. the solution is trivial.

The method:

- Thus, to avoid triviality, k must be zero or negative, and we can write

$$k = -b^2$$

  where b is real or zero, to stress this fact.

- In that case, the solution to the equation for X is, in general,

$$X(x) = A \sin(bx) + B \cos(bx)$$

- Applying the (spatial) boundary conditions then gives

$$B = 0 \qquad \text{and} \qquad A \sin(bL) = 0$$

The method:

- Since we want to avoid trivial solutions,
(so we don't want both A and B to be zero),
we must ask that

$$\sin(bL) = 0,$$

which means that

$$bL = n\pi \qquad \text{where } n = 1, 2, 3, \ldots \text{ is a positive integer.}$$

- That is:

$$b = \frac{n\pi}{L}.$$

- Thence

$$X(x) \propto \sin\left(\frac{n\pi}{L}x\right)$$

The method:

- But this now means that $T(t)$ is very tightly constrained, it must satisfy

$$\frac{T''(t)}{T(t)} = k = -b^2 = -\frac{n^2\pi^2}{L^2}$$

with general solution

$$T(t) = A_n \sin\left(\frac{n\pi t}{L}\right) + B_n \cos\left(\frac{n\pi t}{L}\right).$$

The method:

Thus we have discovered that

$$U(x,t) = X(x)\ T(t) \quad = \quad A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right)$$

is a solution of the wave equation, and the homogeneous boundary conditions, whatever the constants A_n and B_n, and whatever the integer n.

Here we have written the constants A and B suffixed with n,
to stress the fact that they can be different constants for each choice of $k = -b^2 = -n^2\pi^2/L^2$, or equivalently, of the integer n.

The method:

6. Superposition:

By the principle of superposition, any arbitrary linear combination of these solutions is also a solution satisfying the same homogeneous conditions.

Note that this works because the equation is linear and the boundary conditions are homogeneous (i.e., if we had done the above work and found a class of solutions satisfying non-homogeneous conditions, then we could not assert that arbitrary linear combinations of them also satisfy both the equation and the conditions!)

The method:

That is, the function

$$U(x,t) = \sum_{n=1}^{\infty} \left\{ A_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right) \right\}$$

is also a solution of the wave equation,
satisfying the same homogeneous boundary conditions.

We have not yet enforced the inhomogeneous boundary conditions.

The method:

Warning

Note the SOV approach is primarily designed for use on linear PDEs.

If the PDE is not linear, then even if you succeed in separating variables, (difficult at best), you could not now appeal to superposition to construct the general solution.

Though there is a considerable industry of applying SOV to quasi-linear and nonlinear PDEs, that industry is aimed more at finding specific solutions rather than general solutions.

See, for instance, the Polyanin series of handbooks on PDEs for more than you ever wanted to know about "functional separation of variables".

The method:

Thankfully, may of the most important PDEs are linear.
For example:

- wave equation;
- heat equation;
- Laplace's equation;
- many Euler equations;
- and their cousins.

For more complicated nonlinear PDEs such as

- Einstein's equations of general relativity,
- Navier–Stokes equations of fluid mechanics,

the situation is much messier.

The method:

Comment

(And SOV is typically not a useful technique for solving those PDEs — unless you have an awful lot of symmetry in the problem; which means, [by definition], that you cannot be looking for a truly general solution.)

Question

The generalization to non-homogeneous boundary conditions is actually not too difficult.

(Just make sure it's a linear PDE.)

Any ideas?

The method:

7. Fit series:

Now try to fit the series solution you have found to the remaining non-homogeneous conditions.

(Typically, but not always, initial conditions.)

In general you will get something like the Fourier problem,
for which the solution is well known.

(You have not seen Fourier series yet. Fourier series are the next topic.)

The method:

The condition

$$U(x,0) = f(x)$$

gives

$$\sum_{n=0}^{\infty} B_n \, \sin(n\pi x/L) = f(x) \qquad (1)$$

(note that there is effectively no B_0).

The condition

$$U_t(x,0) = g(x)$$

gives

$$\sum_{n=0}^{\infty} A_n \, \frac{n\pi}{L} \sin(n\pi x/L) = g(x). \qquad (2)$$

The method:

Historically this is the way Fourier series were first encountered.

Since $f(x)$ and $g(x)$ above are arbitrary functions of x, and physically we know the vibrating string had better have a mathematical solution for arbitrary initial data, and the SOV technique seems to indicate that $f(x)$ and $g(x)$ must be sums of sines and cosines, this strongly suggests that (more or less) arbitrary functions of x can be represented as sums of sines and cosines.

This is the "miracle" of Fourier series, and at first mathematicians and physicists simply did not believe their own results.

The method:

From Fourier Series theory the constants A_n and B_n can be found in the usual way.

(I will justify these formulae later.)

From (1)

$$A_n = \frac{2}{n\pi} \int_0^L g(x) \sin(n\pi x/L)\, \mathrm{d}x \qquad [n = 1, 2, 3, ...]$$

and from (2) we have

$$B_n = \frac{2}{L} \int_0^L f(x) \sin(n\pi x/L)\, \mathrm{d}x \qquad [n = 1, 2, 3, ...]$$

The method:

So finally the solution satisfying all the given conditions is

$$U(x,t) = \sum_{n=1}^{\infty} \left[A_n \sin(n\pi x/L)\sin(n\pi t/L) + B_n \sin(n\pi x/L)\cos(n\pi t/L)\right]$$

with

$$A_n = \frac{2}{n\pi} \int_0^L g(x)\; \sin(n\pi x/L)\, \mathrm{d}x \qquad [n = 1, 2, 3, ...]$$

and

$$B_n = \frac{2}{L} \int_0^L f(x)\; \sin(n\pi x/L)\, \mathrm{d}x \qquad [n = 1, 2, 3, ...]$$

The method:

8. A specific example:

Suppose the string has length $L = 2$, and was plucked in such a way that

$$f(x) = \begin{cases} x/10 & \text{for } 0 \le x \le 1 \\ (2-x)/10 & \text{for } 1 \le x \le 2 \end{cases}$$

and

$$g(x) = 0$$

corresponding to the string being initially held fixed.

The method:

Then you find

$$A_n = 0 \qquad \text{for all n}$$

and

$$B_n = \frac{4}{5}\frac{\sin(n\pi/2)}{n^2\pi^2}$$

The method:

The explicit solution to our example is

$$U(x,t) = \sum_{n=0}^{\infty} \left\{ \frac{4}{5} \, \frac{\sin(n\pi/2)}{n^2\pi^2} \times \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi t}{L}\right) \right\}$$

which is perhaps not very edifying!

The method:

Note

- $\sin(n\pi/2)$ is non-zero iff $n = 2m + 1$ is odd.
- $\sin(n\pi/2) = \sin([m + 1/2]\pi) = (-1)^m$.

The explicit solution to our example is

$$
\boxed{\begin{aligned} U(x,t) \;=\; & \sum_{m=0}^{\infty} \left\{ \frac{4}{5} \frac{(-1)^m}{(2m+1)^2\pi^2} \right. \\ & \left. \times \sin\left(\left[m + \frac{1}{2} \right] \pi x \right) \cos\left(\left[m + \frac{1}{2} \right] \pi t \right) \right\} \end{aligned}}
$$

which is perhaps not very edifying!

The method:

You can use Maple to plot a diagram of the sum taking, say, the first ten terms (this should give a pretty good picture of the situation).

Exercise

Use Maple to generate a plot, truncated at the 30'th term, for time values $t = 0.0, 0.5, 1.0, 1.5,$ *and* 2.0.

You should see (quite clearly) the development of oscillations in the string.

Note that the rounded edges near peaks are simply due to the truncation.

(More on this below.)

Separation of variables: Comments

Comments on SOV:

Possible complications:

I have presented only the simplest form of the SOV technique.

It can be modified in much more general ways.

For example, for systems where there is some version of spherical symmetry it is often useful to write

$$U(t, x, y, z) = T(t)\; R(r)\; L(\theta)\; \Phi(\phi).$$

Comments on SOV:

If you then consider the $(3+1)$–dimensional wave equation it will separate, but you might be a little surprised at the results

- $T(t)$ is a complex exponential (sine plus cosine)
- $\Phi(\phi)$ is a complex exponential (sine plus cosine)
- $L(\theta)$ is a Legendre polynomial in the variable $\cos\theta$.
- $R(r)$ is a spherical Bessel function.
- The combinations $Y(\theta,\phi) = L(\theta)\,\Phi(\phi)$ are the spherical harmonics.

In other words, sines and cosines often arise in SOV,
but more complicated functions also show up.

Comments on SOV:

If you are solving the wave equation on a circular drum-head,
[two space dimensions, one time dimension],
you will typically get

$$U(t, x, y) = T(t)\, B(r)\, \Phi(\phi).$$

Here:

- $T(t)$ is a complex exponential (sine plus cosine).
- $\Phi(\phi)$ is a complex exponential (sine plus cosine).
- $B(r)$ is an ordinary Bessel function.

Again, sines and cosines often arise in SOV,
but more complicated functions also show up.

Comments on SOV:

Comment

Of course, the fact that Bessel functions (both ordinary and spherical) show up in such simple applications of SOV to the wave equation is the fundamental reason why applied mathematicians are so interested in Bessel functions — this is why they have been given a special name, and why the properties of Bessel functions have been so intensely investigated.

Separation of variables: Sufficient conditions

A sufficient condition:

A sufficient condition for the SOV technique to work is described below:

Consider the linear homogeneous PDE

$$D_n(x^i, \partial_i)\; U_n(x^i) - \lambda_n\; U_n(x^i) = 0,$$

where D_n is some partial differential operator in n independent variables.

Comments on SOV:

IF you can find a coordinate system such that the partial differential operator decomposes into a sum of an ordinary differential operator, plus something proportional to a lower-dimensional partial differential operator,

$$D_n(x^i, \partial_i) = D_1(x^1, \partial_1) + h_1(x^1)\; D_{n-1}(x^{i\neq 1}, \partial_{i\neq 1}),$$

where D_1 is an ordinary differential operator which involves only one of the independent variables, h_1 is a function which depends only on x_1, and D_{n-1} involves the remaining $n-1$ independent variables,
THEN you can begin to apply the SOV technique.

Comments on SOV:

Definition

Partially separable coordinates:
A coordinate system such that

$$D_n(x^i, \partial_i) = D_1(x^1, \partial_1) + h_1(x_1)\; D_{n-1}(x^{i\neq 1}, \partial_{i\neq 1})$$

is said to be partially separable for the partial differential operator D_n.

Comments on SOV:

We want to solve

$$D_n(x^i, \partial_i)\; U_n(x^i) - \lambda_n\; U_n(x^i) = 0,$$

Now take U_n in partially separated form

$$U_n(x_i) = X(x_1)\; U_{n-1}(x_{i\neq 1}),$$

then

$$D_n U_n = (D_1 X)\; U_{n-1} + X\; h_1\; (D_{n-1} U_{n-1}),$$

which implies

$$\frac{D_1 X}{X} + h_1\; \frac{D_{n-1} U_{n-1}}{U_{n-1}} - \lambda_n = 0.$$

That is

$$\left(\frac{D_1 X}{X} - \lambda_n \right) \frac{1}{h_1} + \frac{D_{n-1} U_{n-1}}{U_{n-1}} = 0,$$

with the first term depending only on x_1, and the second term only on the other $n-1$ independent variables $x_{i\neq 1}$.

Comments on SOV:

Therefore there exists a number λ_{n-1}, the separation constant, such that

$$D_{n-1} U_{n-1} = \lambda_{n-1}\; U_{n-1};$$

$$D_1 X = \{\lambda_n - \lambda_{n-1}\; h_1\}\, X.$$

This has reduced the number of independent variables by one,
and split the problem into a simpler PDE in $n-1$ independent variables,
plus a linear ODE.

IF you can iterate this process all the way down to $n = 1$, then the system is completely separable.

(And even if the problem is only partially separable,
that may still represent significant progress.)

Separation of variables: Abstract examples

Examples:

For the Laplacian operator the following coordinate systems are separable:

- Cartesian coordinates

$$\nabla^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 .$$

Comments on SOV:

- Spherical polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Now apply the above analysis to re-write this as:

$$\nabla^2 = \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right] + \frac{1}{r^2}\left[\left\{\frac{\partial^2}{\partial \theta^2} + \cot\theta\frac{\partial}{\partial \theta}\right\} + \frac{1}{\sin^2\theta}\left\{\frac{\partial^2}{\partial \phi^2}\right\}\right]$$

Note it is completely separable.

Comments on SOV:

3-D Laplacian:

- There are eleven other separable coordinate systems known for this problem; some are relatively simple (cylindrical polar coordinates) others are more obscure (prolate and oblate spheroidal coordinates). And some I've never heard of.

Comments on SOV:

3-D Laplacian:

- Cartesian
- Spherical polar
- Cylindrical polar
- Parabolic cylindrical
- Paraboloidal
- Elliptic cylindrical
- Prolate spheroidal
- Oblate spheroidal
- Ellipsoidal
- Bipolar
- Toroidal
- Conical

Comments on SOV:

Boundary Conditions:
To finish applying SOV you will need to verify that in this same coordinate system the boundary conditions "factorize" in the sense that they can be written as independent sets of boundary conditions for each variable x_i that do not "cross communicate" with each other.

Separation of variables: Exercises

SOV Exercises:

Heat equation:

A square copper sheet has its edges maintained at prescribed temperatures.

Along the x and y axes the temperature is held to zero (say by a nice big block of ice).

The temperature is also held to zero along the side given by the line $x = 1$.

Finally, along the fourth edge of the square at $y = 1$ the temperature is held at $100x(1 - x)$ — so that it is zero at the edges and rises quadratically with a maximum of 25 at the centre of this edge.

SOV Exercises:

When all has settled down to equilibrium, the distribution of heat in the slab satisfies Laplace's equation

$$U_{xx} + U_{yy} = 0 \qquad (0 \leq x, y \leq 1).$$

where $U(x, y)$ is the temperature at the point (x, y) in the slab.

From the situation described, we have the boundary conditions:

$$U(x, 0) = 0$$

$$U(0, y) = 0$$

$$U(1, y) = 0$$

$$U(x, 1) = 100x(1 - x)$$

SOV Exercises:

a. Find the distribution of temperature in the slab at equilibrium. Of course, you will try separation of variables: $U = X(x)\ Y(y)$, and deduce that $X'' = -b^2 X$ and $Y'' = +b^2 Y$ where b is real.

b. You will also demonstrate that X cannot satisfy the alternative possibility $X'' = +b^2 X$ for real b.

c. Then you will apply the homogeneous BC to find out about b.

d. An isotherm is a curve of constant temperature.
Sketch the isotherms for temperatures 0, 10, 20, 30 and 40 degrees.
[Here is your chance to use Maple.]

Applying Laplace's general solution:

Have a go at the previous question by noting that the general solution of $U_{xx} + U_{yy} = 0$ is given by

$$U(x, y) = F(x + iy) + G(x - iy)$$

(where F and G are arbitrary functions) and trying to fit this general solution to the given conditions.

Do not be surprised to find that it seems impossible.

Elastic string (from SOV to d'Alembert and back again):

For a finite elastic string stretched between $x = 0$ and $x = L$,
the equation describing its displacement $U(x,t)$ away from
the equilibrium configuration at position x at time t is the wave equation:

$$\frac{\partial^2 U(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 U(x,t)}{\partial t^2} = 0$$

Here c is a constant depending on the elastic properties of the string and its tension.
We shall suppose $c = 1$.

SOV Exercises:

The appropriate boundary conditions for the problem are:

i. BC1:
$U(x, 0) = f(x)$ for $0 < x < L$, describing the initial shape of the string when first plucked.

ii. BC2:
$U_t(x, 0) = 0$ for $0 < x < L$, stating that the string initially was held in the shape of $f(x)$ and was then released from rest.

iii. BC3:
$U(0, t) = 0$, stating that the string is permanently fixed at $x = 0$.

iv. BC4:
$U(L, t) = 0$, stating that the string is also permanently fixed at $x = L$.

When you solve this equation using the method Separation of Variables, you find the solution is of the form:

$$U(x,t) = \sum_{n=0}^{\infty} A_n \, \sin(n\pi x/L) \, \cos(n\pi t/L).$$

Where

$$A_n = \frac{2}{L} \int_0^L f(x) \, \sin(n\pi x/L) \, \mathrm{d}x.$$

You will notice that the solution you have here is defined for all x and t.

For any fixed t it is an odd function for all x which is periodic with period $2L$, and for any fixed x it is an even function which is periodic with period $2L$.

SOV Exercises:

On the other hand, you know that the general solution to the wave equation is $U(x,t) = F(x+t) + G(x-t)$ where F and G are arbitrary.

The boundary conditions then imply that:

$$F(x) + G(x) = f(x) \qquad \text{for } 0 < x < L$$

$$F'(x) - G'(x) = 0 \qquad \text{for } 0 < x < L$$

$$F(t) + G(-t) = 0 \qquad \text{for all } t.$$

$$F(L+t) + G(L-t) = 0 \qquad \text{for all } t.$$

SOV Exercises:

Use these conditions to show that $U(x, t)$ has the properties alluded to above, viz that it is defined for all x and t, for any fixed t it is an odd function for all x which is periodic with period $2L$ and for any fixed x it is an even function which is periodic with period $2L$.

Hence show in general that the solution can be expressed in the form you found using separation of variables.

SOV Exercises:

Heat equation:

Solve the heat equation for diffusion of heat down a bar of length $L = 10$:

$$U_{xx} = \frac{1}{k^2}\, U_t$$

subject to the conditions

$$\begin{aligned} U(x,0) &= x && \text{for } 0 < x < 5 \\ &= 10 - x && \text{for } 5 < x < 10 \\ U(0,t) &= 0 = U(10,t). \end{aligned}$$

Take $k = 1$ for argument's sake.

SOV Exercises:

Graph the distribution of temperature down the bar:

i. Initially.

ii. At time $t = 3$.

(Plot only the first few terms of the Fourier series you should have. Indeed, if you are a Maple fanatic, you could present a rather good time sequence here. Choose a value for k for yourself.)

iii. After an extremely long time.

End: