Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

— MATH 301 — PDEs —
Autumn 2024

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21 February 2024

Outline:

Administrivia

Administrivia:

- Lectures:
  - Monday; 12:00–12:50; MYLT 102.
  - Tuesday; 12:00–12:50; MYLT 220.
  - Friday; 12:00–12:50; MYLT 220.
- Tutorial:
  - Thursday; 12:00–12:50; MYLT 220.
- Lecturers:
  - Part 1: Matt Visser.
  - Part 2: Dimitrios Mitsotakis.

Gibbs phenomenon

Gibbs phenomenon:

- From the various Maple worksheets we have seen, it is clear that the "squiggles", the Gibbs phenomenon, have to do with discontinuities in the function $f(x)$...
- But for the Fourier theorem to apply $f(x)$ must be piecewise continuous...
- Therefore:

$$f(x) = (\text{continuous and periodic}) + (\text{finite number of finite discontinuities})$$

- With regards to the Gibbs phenomenon we need only focus on the:

$$(\text{finite number of finite discontinuities})$$

Gibbs phenomenon:

- But since the process of calculating the Fourier coefficients, and summing the Fourier series is linear, there is no loss of generality in focussing on just a single one of these discontinuities.
- In fact, there is really no loss of generality in considering a step discontinuity:
$$f(x) = \mathrm{signum}(x - a)$$
- For simplicity, (ie, good enough for most purposes), we set $a = 0$, so that we consider
$$f(x) = \mathrm{signum}(x)$$
- We saw, in one of the Maple worksheets, that $a \neq 0$ was qualitatively similar to $a = 0$.

Gibbs phenomenon:

- The function

$$f(x) = \mathrm{signum}(x)$$

  is odd, so the natural thing to do is consider a sine series...
- We might as well work on the unit interval $[-1, +1]$.
- The Fourier coefficients are

$$A_n = 2\int_0^1 1 \cdot \sin(n\pi x)\mathrm{d}x = \frac{2}{n\pi}[-\cos(n\pi x)]_0^1 = \frac{2}{n\pi}\{1 - \cos(n\pi)\}$$

- That is

$$A_{2m} = 0; \qquad A_{2m+1} = \frac{4}{\pi(2m+1)}$$

- Therefore

$$\mathrm{signum}(x) = \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{\sin([2m+1]\pi x)}{2m+1}$$

Gibbs phenomenon:

- Now define the finite sum

$$S_M(x) = \frac{4}{\pi} \sum_{m=0}^{M} \frac{\sin([2m+1]\pi x)}{2m+1}$$

- That is

$$S_M(x) = \frac{4}{\pi} \left\{ \sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots + \frac{\sin([2M+1]\pi x)}{2M+1} \right\}$$

- For each fixed x we have

$$\lim_{M\to\infty} S_M(x) = \mathrm{signum}(x)$$

- This is the content of the Fourier convergence theorem...

- But what else can we say about this series

$$S_M(x) = \frac{4}{\pi}\left\{\sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots + \frac{\sin([2M+1]\pi x)}{2M+1}\right\}$$

- Let's evaluate this sum at the specific M-dependent point

$$x = \frac{1}{2[M+1]}.$$

- Then:

$$S_M\left(\frac{1}{2[M+1]}\right) = \frac{4}{\pi}\sum_{m=0}^{M}\frac{\sin\left(\frac{[2m+1]\pi}{2[M+1]}\right)}{2m+1}$$

- That is:

$$S_M\left(\frac{1}{2[M+1]}\right) = \frac{4}{\pi}\sum_{m=0}^{M}\frac{\sin\left(\frac{[2m+1]\pi}{2[M+1]}\right)}{\frac{2m+1}{2[M+1]}}\,\frac{1}{2[M+1]}$$

- That is:

$$S_M\left(\frac{1}{2[M+1]}\right) = \frac{2}{\pi}\sum_{m=0}^{M} \frac{\sin\left(\frac{[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}} \frac{1}{M+1}$$

- But note that the sum is just the mid-point Riemann sum for approximating the integral

$$\sum_{m=0}^{M} \frac{\sin\left(\frac{[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}} \frac{1}{M+1} \approx \int_0^1 \frac{\sin(\pi u)}{u}\mathrm{d}u$$

- Note that $\sin(\pi u)/u$ is continuous...
- So it is certainly Riemann integrable...

Gibbs phenomenon:

- Therefore the limit $M \to \infty$ exists, and we have:

$$\lim_{M\to\infty} S_M\left(\frac{1}{2[M+1]}\right) = \frac{2}{\pi}\int_0^1 \frac{\sin(\pi u)}{u}\mathrm{d}u = \frac{2\,\mathrm{Si}(\pi)}{\pi}$$

- (There are many other ways of getting to the same conclusion.)
- Numerically:

$$\lim_{M\to\infty} S_M\left(\frac{1}{2[M+1]}\right) = \frac{2\,\mathrm{Si}(\pi)}{\pi} = 1.178979744 > 1.$$

- So there is guaranteed to be an overshoot...
- Since the gap from –1 to $+1$ is 2, the fractional overshoot is

$$\Delta = \frac{\frac{2\,\mathrm{Si}(\pi)}{\pi} - 1}{2} = \frac{\mathrm{Si}(\pi)}{\pi} - \frac{1}{2} = 0.0894898720 \approx 9\%$$

Gibbs phenomenon:

- This 9% overshoot is the Gibbs phenomenon...
- (It should really be called the Wilbraham phenomenon.)
- Let us now consider a slightly more general idea:

$$S_M\left(\frac{w}{2[M+1]}\right) = \frac{4}{\pi}\sum_{m=0}^{M} \frac{\sin\left(\frac{w[2m+1]\pi}{2[M+1]}\right)}{2m+1}$$

- Repeating the analysis (with trivial modifications) we see:

$$S_M\left(\frac{w}{2[M+1]}\right) = \frac{2}{\pi}\sum_{m=0}^{M} \frac{\sin\left(\frac{w[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}}\frac{1}{M+1}$$

Gibbs phenomenon:

- But note that the sum is just the mid-point Riemann sum for approximating the integral

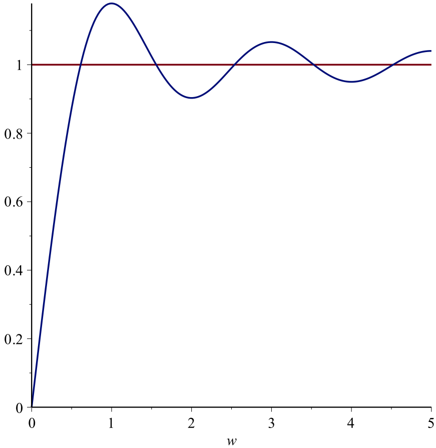
$$\sum_{m=0}^{M} \frac{\sin\left(\frac{w[m+\frac{1}{2}]\pi}{M+1}\right)}{\frac{m+\frac{1}{2}}{M+1}} \frac{1}{M+1} \approx \int_0^1 \frac{\sin(w\pi u)}{u} \mathrm{d}u$$

- Therefore:

$$\lim_{M\to\infty} S_M\left(\frac{w}{2[M+1]}\right) = \frac{2}{\pi}\int_0^1 \frac{\sin(w\pi u)}{u}\mathrm{d}u = \frac{2\,\mathrm{Si}(w\pi)}{\pi}$$

- (There are many other ways of getting to the same conclusion.)

Gibbs phenomenon:



$$\lim_{M\to\infty} S_M\left(\frac{w}{2[M+1]}\right) = \frac{2\,\mathrm{Si}(w\pi)}{\pi}$$

Gibbs phenomenon:

- That is, near a discontinuity we have:

$$S_M(x) \approx \frac{2\ \mathrm{Si}(2\pi x[M+1])}{\pi}$$

- This is actually a reasonably good approximation,
  at least as long as you are closer to the discontinuity at $x = 0$
  than you are to the other discontinuity at $x = \pm 1$.

Gibbs phenomenon:

- We can also argue as follows:

$$S_M(x) = \frac{4}{\pi} \sum_{m=0}^{M} \frac{\sin([2m+1]\pi x)}{2m+1} = 4 \int_0^x \sum_{m=0}^{M} \cos([2m+1]\pi u) du$$

- Then performing the sum

$$S_M(x) = 2 \int_0^x \frac{\sin([2M+2]\pi u)}{\sin(\pi u)} du$$

- For $|x| \ll 1$ we have $|u| < |x| \ll 1$ so $\sin(\pi u) \approx \pi u$ and

$$S_M(x) \approx \frac{2}{\pi} \int_0^x \frac{\sin([2M+2]\pi u)}{u} du$$

Gibbs phenomenon:

- Change variables:

$$S_M(x) \approx \frac{2}{\pi} \int_0^{[2M+2]\pi x} \frac{\sin u}{u} du$$

- That is

$$S_M(x) \approx \frac{2}{\pi} \operatorname{Si}(2[M+1]\pi x)$$

  as before...

- This is the Gibbs phenomenon, generic to discontinuous functions.
- Similar things happen for the sawtooth function.

End: