MATH 301 **Tutorial 1**

1. Identify the dependent and independent variables, and give the order of the following equations:
   (a) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$
   (b) $u_{xx}v_{yy} = \cos(x+y)$
   (c) $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 u}{\partial x^2}$
   (d) $u_t = v_{xxx} + v(1-v), \ v_t = u_{xxy} + vw, \ w_t = u_x + v_y$

Solution:

(a) (Dependent; independent; order) $u, v; x, y; 1$

(b) $u, v; x, y; 2$

(c) $u, v; t, x; 2$

(d) $u, v, w; t, x, y; 3$

2. Give the order and classify the following equations as (i) homogeneous linear, (ii) inhomogeneous linear, or (iii) nonlinear:
   (a) $u_t = x^2 u_{xx} + 2xu_x$
   (b) $e^y u_x = e^x u_y$
   (c) $u_x + e^y u_y = 0$
   (d) $u_{xx} + \cos(xy)u_{xxy} = u + \ln(x^2 + y^2)$
   (e) $x^2 u_{yy} - yuu_x = u$

Solution:

(a) 2; homogeneous linear

(b) 1; homogeneous linear

(c) 1; homogeneous linear

(d) 3; inhomogeneous linear

(e) 2; homogeneous nonlinear

3. Solve the PDE $4u_t - 3u_x = 0$ with initial condition $u(x, 0) = x^3$.

Solution: Based on directional derivative, $u(t,x) = f(-3t-4x)$. IC gives $x^3 = f(-4x)$. So $f(x) = -x^3/64$, and therefore $u(t,x) = (3t+4x)^3/64$.

4. Find the general solution to the initial value problem $u_t + u_x = 0$, $u(1,x) = x/(1+x^2)$

Solution: $u(t,x) = \frac{1-t+x}{(t-x)(2-t-x)}$. If you are feeling masochistic, differentiate it wrt t and x to check that it works.

5. Solve the equation $(1+t^2)u_t + u_x = 0$.

Solution: Characteristics are $dx/dt = 1/(1+t^2)$ (after rewriting the equation). Hence $x = \tan^{-1} t + \xi$ and so $u = f(x - \tan^{-1} t)$

6. Consider the equation $u_x + yu_y = 0$, $u(x,0) = f(x)$. Show that there is no solution for $f(x) \equiv x$, but many for $f(x) \equiv 1$.

Solution: Characteristic curves are $dy/dx = y$, which (by separation of variables) is $y = \xi e^x$, so $u = f(ye^{-x})$. When $f(x) \equiv x$ $u(x,0) = f(x) = x$. But $u = x$ doesn't solve the PDE, so there are no solutions. For the second part $u = 1$ does solve the PDE, but it doesn't help us specify what the function f is anywhere else.

7. Apply the method of characteristics to the problem $u_t + c(u)u_x = 0$ for $x \in (-\infty, +\infty)$ and $t \geq 0$, where $c'(u) > 0$ and $u(0,x) = f(x)$. Find a formula for the breaking time.

Solution: The characteristic lines will be $x = c(f(\xi))t + \xi$. The breaking time is given by the formula

$$t_b = \min_{\xi} \frac{-1}{f'(\xi)c'(f(\xi))}, \quad t_b > 0.$$

8. Consider the wave equation

$$u_{tt} = c^2 u_{xx} \ .$$

First observe that the wave equation is a linear equation. Prove that if $u_1(t,x)$ and $u_2(t,x)$ are two solutions of the wave equation then $u_3(x,t) = u_1(x,t) + u_2(x,t)$ is also a solution of the wave equation. This is known as *superposition principle* and applies to all linear differential equations.

Solution: Since u_1 and u_2 are solution to the wave equation they satisfy

$$(u_1)_{tt} = c^2 (u_1)_{xx} \ ,$$

and

$$(u_2)_{tt} = c^2(u_2)_{xx} \; .$$

Then we show that u_3 satisfies also the same equation:

$$\begin{aligned}(u_3)_{tt} &= (u_1+u_2)_{tt}\\ &= (u_1)_{tt} + (u_2)_{tt}\\ &= c^2(u_1)_{xx} + c^2(u_2)_{xx}\\ &= c^2(u_1+u_2)_{xx}\\ &= c^2(u_3)_{xx} \; .\end{aligned}$$

9. Consider the equation $u_t + uu_x = 0$ for $x \in [0,\infty)$ and $t \geq 0$. Assume that the initial condition is $u(0,x) = f(x) > 0$ is very smooth with $f'(x) < 0$ and $\lim_{x\to\infty} f(x) = 0$ exponentially fast.

   (a) How many boundary condition we need in order to determine a solution in $[0,\infty)$.

   (b) Will the solution be unique and up to what time can you guarantee the existence of a unique continuous solution?

   (c) What happens if $f(x) < 0$ and $f'(x) > 0$ with $\lim_{x\to\infty} f(x) = 0$?

Solution: Because the characteristics start from $x = 0$ we will need one boundary condition $u(0,t)$ for $t \geq 0$. The solution will be unique and continuous until the breaking time t_b. This is because $f'(x) < 0$ and thus $x_1'(t) = u(0,\xi_1) < u(0,\xi_2) = x_2'(t)$ where $x_1(t)$ and $x_2(t)$ are two characteristic lines starting at $\xi_1 < \xi_2$, respectively. To prove uniqueness assume for contradiction that there are two solutions u_1, u_2 and use the energy method for the difference $U = u_1 - u_2$: Note that $U(0,t) = 0$ and also $U(x,0) = 0$. You will have

$$\begin{aligned} U_t + [U(u_1+u_2)]_x &= 0\\ \text{(multiply by } U\text{)} \int_0^\infty UU_t \; dx - \int_0^\infty UU_x(u_1+u_2)\; dx &= 0\\ \frac{1}{2}\frac{d}{dt}\int_0^\infty U^2 \; dx &= \frac{1}{2}\int_0^\infty (U^2)_x(u_1+u_2)\; dx\\ (u_1+u_2 \leq C)\frac{d}{dt}\int_0^\infty U^2 \; dx &\leq C\int_0^\infty (U^2)_x \; dx\\ \frac{d}{dt}\int_0^\infty U^2 \; dx &\leq 0\\ \frac{d}{dt}\int_0^\infty U^2 \; dx &= 0\end{aligned}$$

Therefore $U \equiv 0$ and thus $u_1 = u_2$.