Name: Model Solutions ID number:

- Duration: 50 MINUTES. 50 Marks
- There are FIVE questions, on FIVE pages. Attempt every question in the spaces provided. Use the reverse side if you run out of space.
- Write your name and ID number on the first page, and clearly label each question attempt.

Question 1. (10 marks)

(a) State the Handshaking Lemma. [2]

For a graph $G=(V,E)$, $\sum_{v\in V} d(v) = 2|E|$. ✓✓

(b) Give the definition of a *forest*. [1]

A forest is a graph with no cycles. ✓

(c) Let G be a connected graph. Give the definition of a *spanning tree* of G. [2]

A spanning tree of G is a subgraph H of G that is a tree ✓

and $|V(H)| = |V(G)|$. ✓

(d) Let G be a non-empty graph that is connected but not a tree. Prove that there exists an edge e in G such that $G\backslash e$ is connected. [5]

Since G is connected, it has a spanning tree H.

By definition, $V(H) = V(G)$, and $E(H) \subseteq E(G)$.

But $H \neq G$, since H is a tree but G is not.

Therefore, there exists an edge $e \in E(G) \setminus E(H)$.

Now $G\backslash e$ has H as a spanning tree, so $G\backslash e$ is connected.

✓✓✓✓✓

Question 2. (11 marks)

(a) Let G be a graph. Give the definition of a *cut vertex* of G. [2]

A vertex $v \in V(G)$ is a cut vertex if $G - v$ has more components than G.

(b) Let G be a graph. Are the following statements true or false? Justify your answer with an explanation if true, or give a counterexample if false.

(i) If G is isomorphic to the path graph P_n for some positive integer n, then G is bipartite. [3]

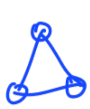
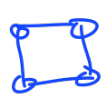
True. Let $v_1, v_2, \dots, v_n$ be the vertices of P_n such that $v_i v_{i+1}$ is an edge for each $i \in \{1, 2, \dots, n-1\}$.
Then letting $A = \{v_i : i \text{ is odd}\}$ and
$B = \{v_i : i \text{ is even}\}$ gives a bipartition of $V(G)$.

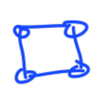
(ii) If G is a 3-connected graph, then G is 2-connected. [3]

True. If G is 3-connected, then $|V(G)| \geq 4$, G is connected and G has no vertex cuts of size 1 or 2. In particular, $|V(G)| \geq 3$ and G has no cut vertices, so G is 2-connected.

(iii) If H is an minor of G, then H is a subgraph of G. [3]

False. Counterexample:

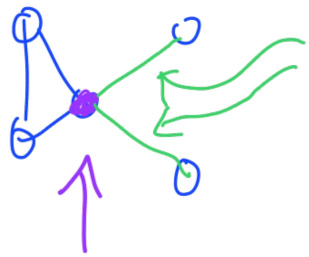
 is a minor of 

(but  is not a subgraph of ).

Question 3. (9 marks)

By drawing an appropriate graph, give a clearly illustrated example of the following:

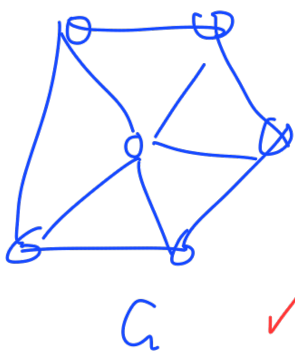
(a) a graph with exactly one cut vertex and exactly two bridges. [3]



the only two bridges.

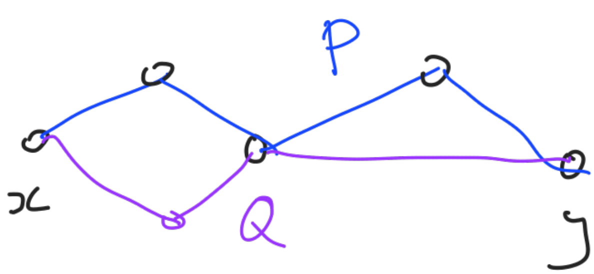
the unique cut vertex

(b) a 3-connected graph G such that, for every edge e of G, the graph $G\backslash e$ is not 3-connected. [3]



Here, for any $e \in E(G)$, the graph $G\backslash e$ has a vertex of degree 2, so $G\backslash e$ is not 3-connected.

(c) a graph with two (x, y)-paths P and Q such that P and Q are edge-disjoint, but not internally vertex-disjoint. [3]



Question 4. (15 marks)

(a) Are the following statements true or false? Justify your answer with an explanation if true, or give a counterexample if false. In the former case, you may refer to results seen in class, without giving a proof.

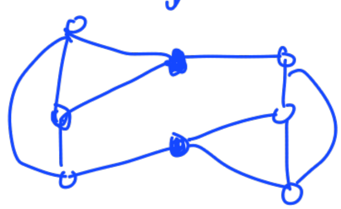
(i) If G is a 3-connected graph with at least five vertices, then there exists an edge e in G such that G/e is 3-connected. [2]

True. ✓ We proved this as a theorem in lectures. ✓

(ii) If G is a non-empty simple graph where every vertex has degree three, then G is 3-connected. [3]

False ✓ e.g.

2-vertex cut, so not 3-connected.



✓✓

(b) Let G be a graph and let X and Y be non-empty subsets of $V(G)$.

(i) Define what is meant by an (X, Y)*-path*. [1]

A path from a vertex $x \in X$ to a vertex $y \in Y$. ✓

(ii) For a set $S \subseteq V(G)$, define what it means for S to *separate* X *from* Y. [1]

S separates X from Y if every (X,Y)-path contains some vertex in S. ✓

(iii) State Menger's theorem. [2]

The minimum size of a set that separates X from Y equals the maximum number of vertex-disjoint (X,Y)-paths. ✓✓

(iv) Explain why it follows from Menger's theorem that if G is 2-connected and X and Y each have size two, then there are two vertex-disjoint (X, Y)-paths in G. [6]

By Menger's theorem, it suffices to show the minimum size of a set that separates X from Y is at least 2.

So suppose S separates X from Y. Since G is connected, $S \neq \emptyset$. If $|S|=1$, then every (X,Y)-path passes through the single vertex s in S. Since $X \setminus S$ and $Y \setminus S$ are non-empty, $G-s$ is disconnected (as a vertex in $X \setminus S$ is in a different component to a vertex in $Y \setminus S$). So s is a cut vertex, contradicting that G is 2-connected. So $|S| \geq 2$ as required.

Question 5. (5 marks)

(a) Define a *plane graph* (you may make reference to a *planar embedding* without defining this term). [1]

A plane graph is a graph together with a planar embedding.

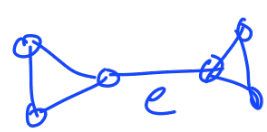
(b) Let G be a plane graph. Define what it means for an edge of G to be *incident* to a face of G. [1]

An edge e of G is incident to a face f if e is in the boundary of f.

(c) Is the following statement true or false? Justify your answer with an explanation if true, or give a counterexample if false.

- Every edge in a plane graph is incident with two distinct faces. [3]

False e.g. the outer face is the only face incident with the edge e.



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