Victoria University of Wellington
School of Mathematics and Statistics

MATH 361	Assignment 3	T1 2024

Due 3pm Tuesday 30 April

1. For a graph G with vertex set V, and disjoint sets $A, B \subseteq V$, let $e(A, B)$ be the number of edges of G with one end in A and the other end in B, and let $d(A) = e(A, V \setminus A)$. Prove, for any $X, Y \subseteq V$, that

$$d(X) + d(Y) \geq d(X \cup Y) + d(X \cap Y).$$

(Hint: one option is to show that $d(X)+d(Y) = d(X\cup Y)+d(X\cap Y)+2e(X\setminus Y, Y\setminus X)$.)

2. Let K_5' be the graph obtained by deleting an edge from K_5.

(a) Draw a planar embedding of K_5'. Clearly label the edges.
(b) How many faces does your embedding have? For each face, list the edges in the boundary of that face.
(c) Now choose a face that is *not* the outer face and draw another planar embedding of G in which that face is the outer face.

3. The Petersen Graph is illustrated in Tutorial Q3, with five edge crossings. Find a drawing of the Petersen Graph on the plane with only two edge crossings.

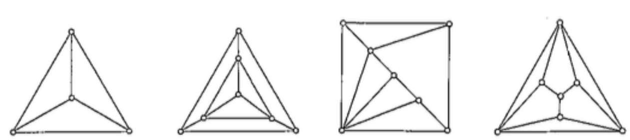
4. Show that K_5 can be embedded on the torus. (Hint: see Tutorial Q4 before attempting this question.)

5. (Exercise 4.4 or Corollary 5.8) By any method, prove that K_5 is not planar.

6. (Exercise 5.9) Prove that $K_{3,3}$ is not planar using Euler's formula. (Hint: use together with the Handshaking Lemma for faces.)

7. A plane graph is *self-dual* if it is isomorphic to its dual.

(a) Show that if G is self dual, then $|E(G)| = 2|V(G)| - 2$.
(b) The four plane graphs illustrated below are self-dual. Each belongs to an infinite family of self-dual plane graphs. Describe one of these infinite families.



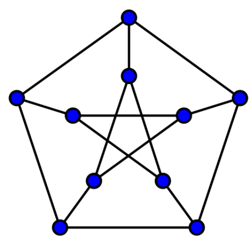
Tutorial exercises:

1. Let G be a plane graph. Prove that if e is a loop in G, then e^* is a bridge in G^*.

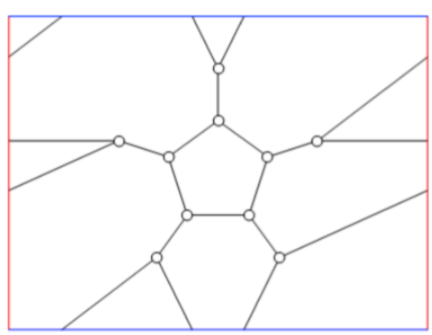
2. Let $K'_{3,3}$ be the graph obtained by deleting an edge from $K_{3,3}$.

(a) Draw a planar embedding of $K'_{3,3}$. Clearly label the edges.
(b) How many faces does your embedding have? For each face, list the edges in the boundary of that face.
(c) Now choose a face that is *not* the outer face and draw another planar embedding of G in which that face is the outer face.

3. Prove that the Petersen Graph has a K_5- or a $K_{3,3}$-minor. Can you find both?



4. For this question, we'll think about graphs embedded on the torus (rather than on the plane). The nice thing is that, while drawing graphs on actual tori can be difficult, we can model them by drawing graphs on rectangles. We just have to imagine that each point at the top of the rectangle matches with the corresponding point at the bottom of the rectangle and the same for points at the side. The Petersen Graph is not planar, but below is an illustration of an embedding of the Petersen Graph on the torus.



Show that $K_{3,3}$ can be embedded on the torus by finding a drawing on the rectangle as described above.

5. For a graph $G = (V, E)$, and a set $A \subseteq E(G)$, let $V(A)$ denote the set of vertices incident with A. Define $\lambda(A)$ as follows:

$$\lambda(A) = |V(A)| + |V(E \setminus A)| - |V|$$

for any $A \subseteq E(G)$. The function λ is called the *connectivity function* of G.

(a) Prove that if G has no isolated vertices, then $\lambda(A)$ counts the number of vertices incident with edges in both A and $E \setminus A$.

(b) Prove that for all $A, B \subseteq E(G)$, the following inequality holds:

$$\lambda(A) + \lambda(B) \geq \lambda(A \cap B) + \lambda(A \cup B).$$

6. This question concerns inequivalent planar embeddings. In particular, we find solutions to Exercises 5.1 and 5.2.

(a) Let G be a 3-connected planar graph with a parallel pair of edges. Explain, with an example, how G may have inequivalent planar embeddings.

(b) Give an example of a simple 2-connected planar graph with inequivalent planar embeddings. Be sure to provide two inequivalent planar embeddings of the graph.

7. In this question, the goal is to work towards a proof of Theorem 5.3: "Let G be a simple 3-connected planar graph. Then G has a unique planar embedding."

(a) The natural strategy is to use induction on the number of vertices of G. After having established a base case we proceed to the next step. Let G be a simple 3-connected planar graph. We know that G has an edge e such that G/e is 3-connected. If G/e is also simple then, by induction, G/e has a unique planar embedding.

Let $e = rb$ be an edge of a simple 3-connected graph G. Let w be the vertex that replaces rb in G/e. Prove that G/e is loopless, has no parallel triples of edges, and that all parallel pairs of edges in G/e are incident with w.

(b) Say the graph G/e is simple. Assume that we know, from our inductive hypothesis, that all simple 3-connected planar graphs with one fewer vertex than G have unique planar embeddings. Explain why G has a unique planar embedding.

(c) Suppose that G/e has a parallel pair. Explain why G has a unique planar embedding.