MATH361 Lecture 1

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office hour Tues 4:10-5

Course info:

Lectures 3:10-4 Mon-Wed

Tutorial 3:10-4 Fri

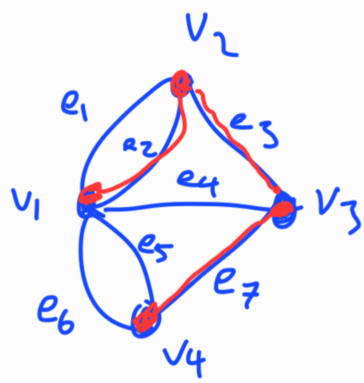
Assessment

5 assignments, 6% each

term test 20%

final test 50%

Graph theory



G

Formally, a graph is:

a set of vertices V,

a set of edges E,

and an incidence function $\varphi : E \to 2^V$

that maps each edge $e \in E$ to a set of 1 or 2 vertices.

e.g. Let $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_1, e_2, \ldots, e_7\}$

the incidence function φ for G is defined as follows

$\varphi(e_1) = \varphi(e_2) = \{v_1, v_2\}$

$\varphi(e_3) = \{v_2, v_3\}$

$\varphi(e_4) = \{v_1, v_3\}$

$\varphi(e_5) = \varphi(e_6) = \{v_1, v_4\}$

$\varphi(e_7) = \{v_3, v_4\}$

We say $v \in V$ is incident with $e \in E$ if $v \in \varphi(e)$

We say $e \in E$ is a loop if $|\varphi(e)| = 1$

$\varphi(e) = v$

We write $G = (V, E)$ to denote a graph on vertex set V and edge set E (with an implicit incidence function).

We say $v_1, v_2 \in V$ are adjacent if there exists some $e \in E$ that is incident with v_1 and v_2.

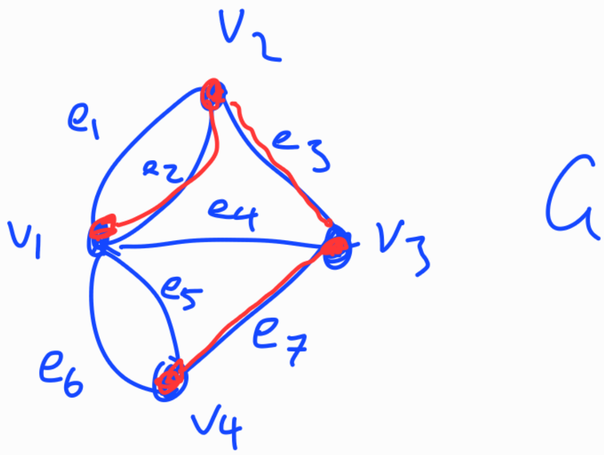
A walk in G is a sequence

$$v_1, e_1, v_2, e_2, \dots, v_t$$

alternating between vertices and edges such that v_i and v_{i+1} are incident with e_i for each $i \in \{1, \dots, t\}$.

e.g.

$v_4, e_7, v_3, e_3, v_2, e_2, v_1$



A path is a walk where no vertex appears more than once.

The degree of a vertex is the number of edges incident with it, where each loop counts for 2.

e.g.

here each vertex has degree 3.

Note: the graph G above represents the 7 bridges of Königsberg.

Recall that an Eulerian walk in a graph H is a walk that traverses each edge of H exactly once.

A graph H has an Eulerian walk if and only if it has 0 or 2 vertices of odd degree.

The graph G has no Eulerian walk since it has 4 vertices of odd degree.

Let $G=(V,E)$ be a graph.
We say that distinct non-loop edges $e_1, e_2 \in E$ are parallel if e_1 and e_2 are incident to the same pair of vertices.

A graph is simple if it has no loops and no parallel edges.

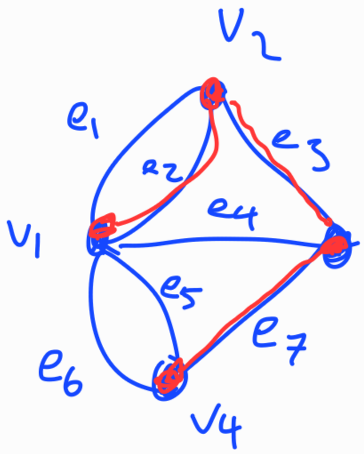
For simple graphs, each edge is determined by its 2 ends. We will say "$e=uv$" as a shorthand for "e is incident with u and v".

The Handshaking Lemma:
We write $d(v)$ to denote the degree of a vertex v.

Theorem 1.3: Let $G=(V,E)$ be a graph.
Then $\sum_{v\in V} d(v) = 2|E|$.

Let $G=(V,E)$ be a graph. The incidence matrix of G has rows labelled by V, columns labelled by E, and, for $v\in V$ and $e\in E$, the entry in the row labelled v and column labelled e is $\begin{cases} 1 & \text{if } v \text{ is incident with } e \\ 0 & \text{otherwise.} \end{cases}$

e.g. G



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

This is the incidence matrix of G.

<u>Proof</u> of Thm 1.3:

Consider the incidence matrix of G. If G has loops we first modify the matrix so that any column corresponding to a loop has a 2 as its non-zero entry.

Now the entries of each column sum to 2. Since there are $|E|$ columns, the sum of all entries is $2|E|$.

Now we sum the entries row by row.

Each row corresponds to a vertex v, and the sum of that row is the degree of the vertex v, $d(v)$. So we get $\sum_{v \in V} d(v)$

Hence $\sum_{v \in V} d(v) = 2|E|$. $\square$