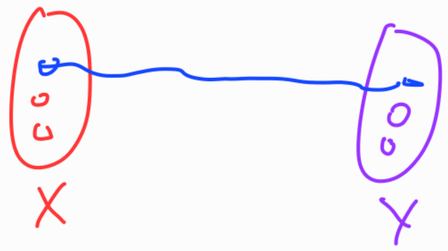
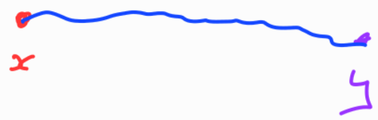
MATH361 | Lecture 11

Last time: Thm 3.21: Let G be a 3-connected graph (with $|V(G)| \geq 5$). Then there exists an edge e such that G/e is 3-connected.

Terminology: (X, Y)-path

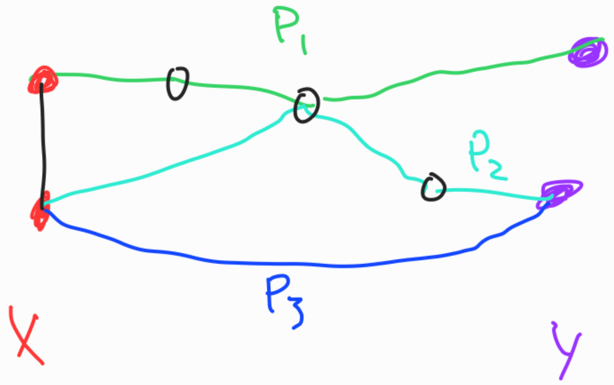


(x, y)-path



vertex-disjoint } path
edge-disjoint }

e.g.



P_1, P_2 and P_3 are (X, Y)-paths

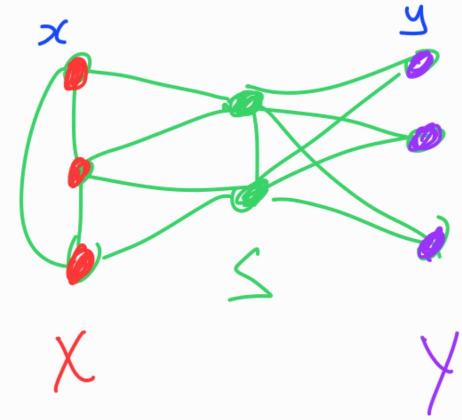
P_1 and P_2 are not vertex-disjoint but they are edge-disjoint.

P_1 and P_3 are both edge-disjoint and vertex-disjoint.

Let $X, Y \subseteq V(G)$ and pick a positive integer k.
When is it possible to find k (pairwise) vertex-disjoint (X, Y)-paths?

Let $X, Y \subseteq V(G)$ be non-empty. For $S \subseteq V(G)$, we say S separates X from Y if every (X, Y)-path contains some vertex in S.

e.g.



S separates X from Y.

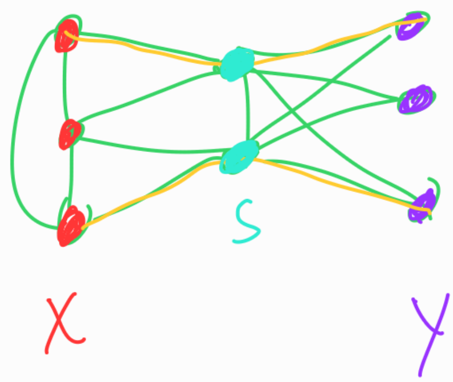
S separates {x} from {y}.

Theorem 3.28 (Menger, 1927):

Let G be a graph, and $X, Y \subseteq V(G)$ are non-empty.

The minimum size of a set that separates X from Y equals the maximum number of vertex-disjoint (X, Y)-paths.

e.g.

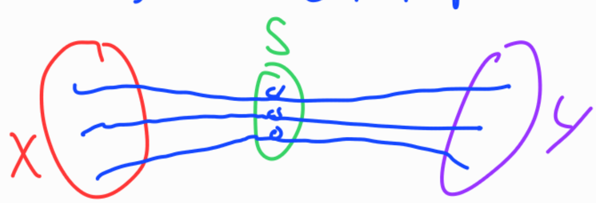


Here S is a set of size 2 that separates X from Y, and there is no smaller set separating X from Y.

There are 2 (but not more than 2) vertex-disjoint (X, Y)-paths.

Lemma 3.27: Let G be a graph. Let $X, Y \subseteq V(G)$ be non-empty, and $S \subseteq V(G)$ where S separates X from Y. Then there are at most $|S|$ vertex-disjoint (X, Y)-paths.

Proof: Since S separates X from Y,

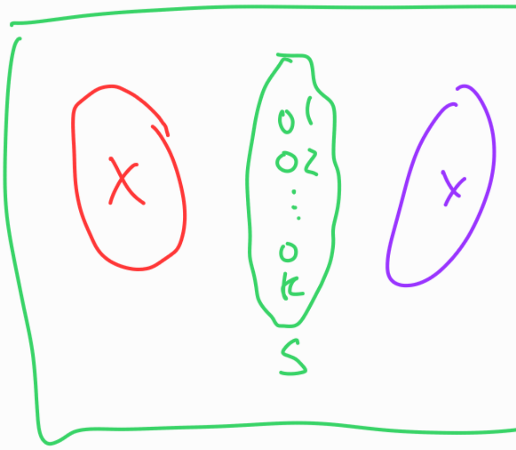


every (X,Y)-path contains a vertex of S. No two vertex-disjoint (X,Y)-paths can contain the same vertex of S. So there are at most $|S|$ vertex-disjoint (X,Y)-paths. □

Proof of Menger's theorem:

Let G be a graph, and $X, Y \subseteq V(G)$ are non-empty. The minimum size of a set that separates X from Y equals the maximum number of vertex-disjoint (X,Y)-paths.

Let S be a set that separates X from Y, where S has minimum size amongst all such sets. Say $|S| = k$.



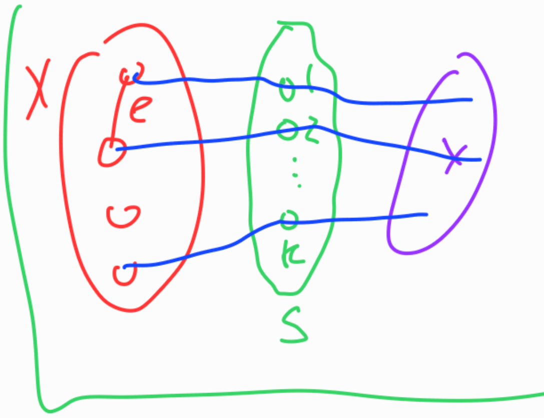
Lemma 3.27 tells us that the number of vertex-disjoint (X,Y)-paths is at most k. We still need to show that there are in fact k such paths. (i.e. not less than k.)

We prove this by induction on the number of edges of G.

It is not too difficult to check the base case where $|E(G)| = 0$ — details omitted.

Now assume $|E(G)| \geq 1$ and the theorem holds for graphs with fewer than $|E(G)|$ edges. (the induction assumption)

Claim: If G has an edge $e = uv$ with $\{u,v\} \subseteq X$ or $\{u,v\} \subseteq Y$, then there are k vertex-disjoint (X,Y)-paths in G.



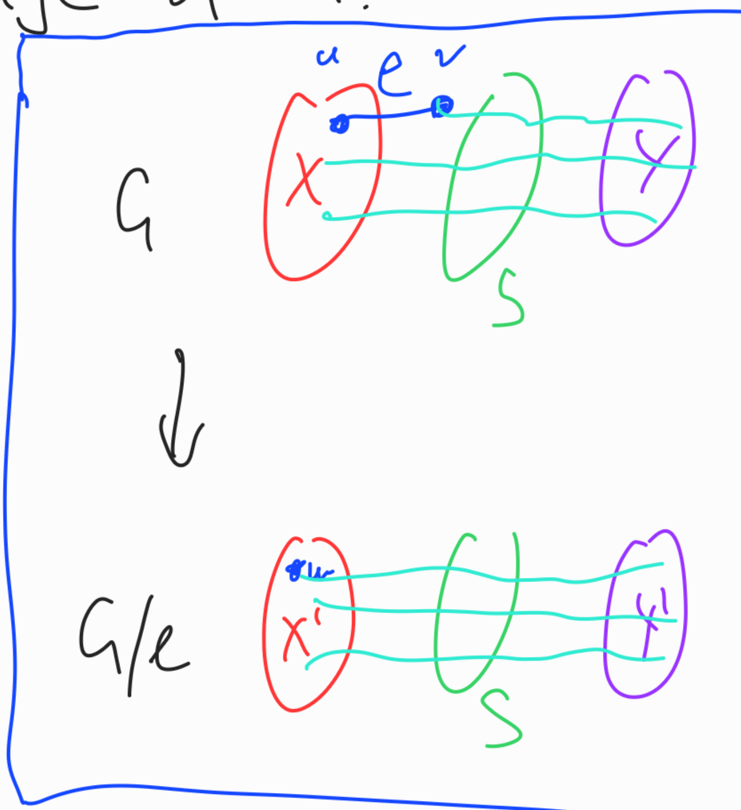
Proof: Suppose Z separates X from Y in G/e. Then Z separates X from Y in G.

So $|Z| \geq |S| = k$.

We apply the induction assumption on G/e and see that there are $|Z|$ vertex-disjoint (X,Y)-paths in G/e, with $|Z| \geq k$, and therefore also k vertex-disjoint (X,Y)-paths in G.

By the claim, we may now assume that G has no edges where both ends are in X or both ends are in Y.

Let $e = uv$ be an edge of G.



Consider G/e where w is the vertex resulting from the contraction of e. Let

X', Y' be obtained from X, Y by replacing u or v with w.

If G/e has k vertex-disjoint (X', Y')-paths, then G has k vertex-disjoint (X, Y)-paths.

Final part of argument sketched in lecture. Will recap tomorrow.