MATH361

Assignment 2 due Tuesday 3pm

Last time: (Thm 3.28):

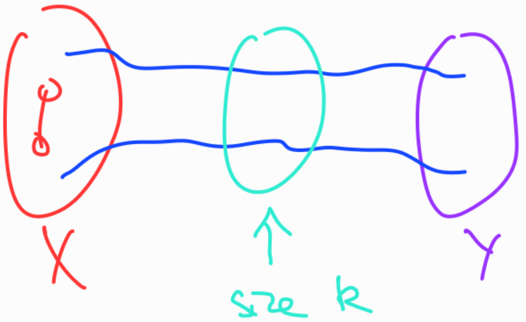
Let G be a graph and let $X, Y \subseteq V(G)$ be non-empty.
The minimum size of a set that separates X from Y is equal to the maximum number of vertex-disjoint (X,Y)-paths.

Proof continued: Recall, induction on $|E(G)|$. $|E(G)| \geq 1$.

k is the minimum size of a set that separates X from Y

We'd proved

Claim: If G has an edge $e = uv$ with $\{u,v\} \subseteq X$ or $\{u,v\} \subseteq Y$, then there are k vertex-disjoint (X,Y)-paths.

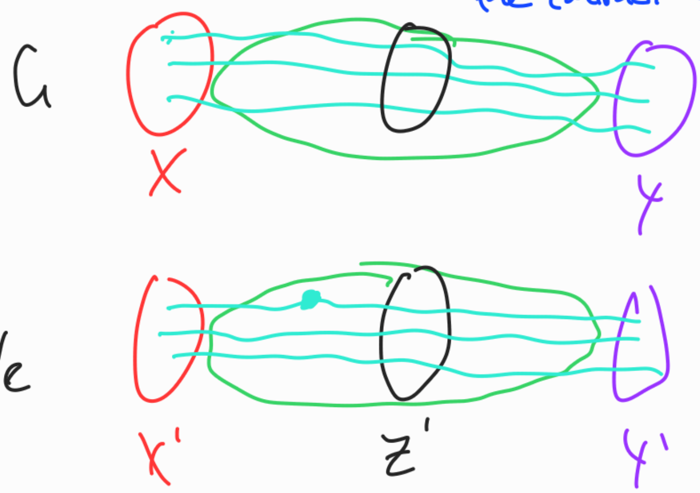


We want to show there are k vertex-disjoint (X,Y)-paths.

We'd picked $e = uv$, and considered G/e (where w is the vertex from the contraction).

Suppose Z' is a set that separates X' from Y' in G/e having minimum size.

By induction, there are $|Z'|$ vertex-disjoint (X', Y')-paths.



If $|Z'| = k$, then there are k vertex-disjoint (X,Y)-paths in G.

Suppose $|Z'| < k$.

Case 1: $w \notin Z'$. Then Z' also separates X from Y in G, so

$|Z'| \geq k$, a contradiction.

Case 2: $w \in Z'$.

Let $Z = (Z' \setminus \{v\}) \cup \{u, v\}$

Then Z separates from Y in G, so

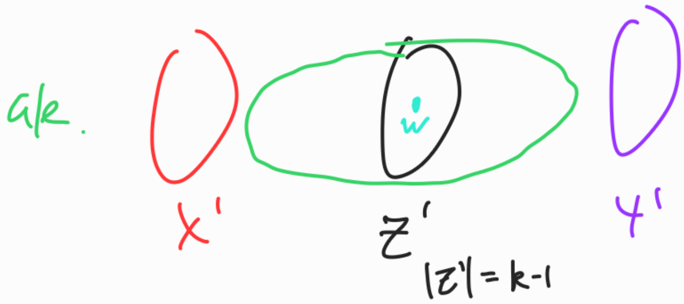
$|Z| = k$.

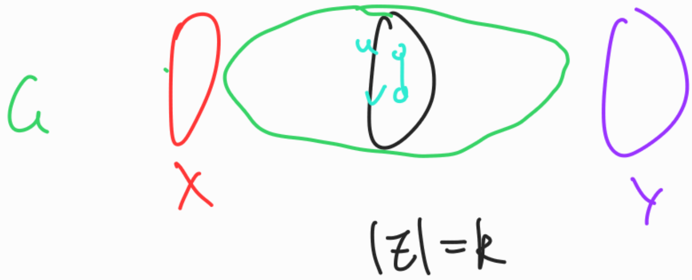
Now, a set that separates X from Z has size at least k (as it would also separate X from Y).

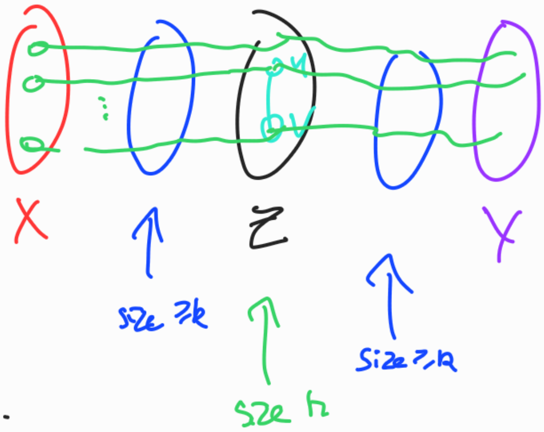
By the earlier claim there are k vertex-disjoint (X, Z)-paths, since $\{u, v\} \subseteq Z$.

Similarly, there are k vertex-disjoint (Z, Y)-paths.

Since $|Z| = k$, by combining these we obtain k vertex-disjoint (X, Y)-paths, as required. $\square$







Answering the question from yesterday:

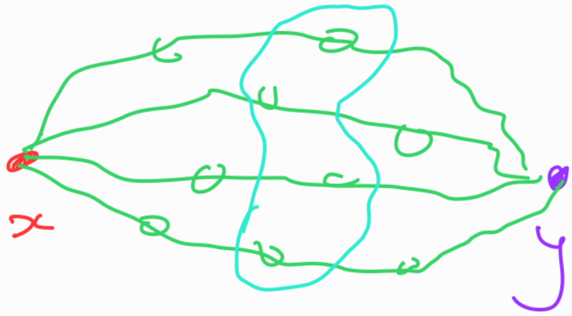
$\boxed{\text{there are } k \text{ vertex-disjoint } (X,Y)\text{-paths in } G \text{ if and only if there is no set of size smaller than } k \text{ that separates } X \text{ from } Y.}$

(Corollary 3.29)

Let x and y be distinct vertices in a graph G.

Two (x, y)-paths P and Q

are internally disjoint if

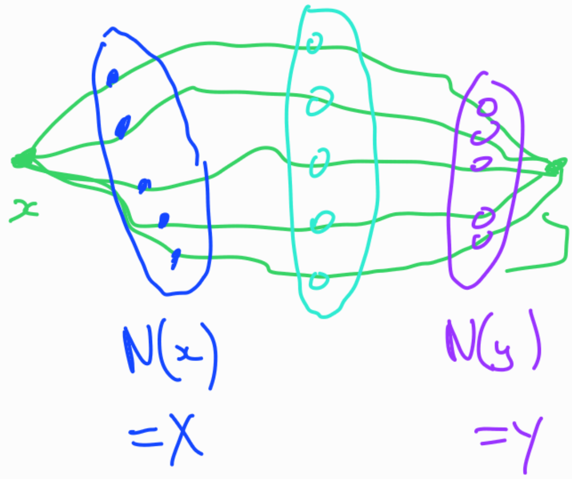


$V(P) \cap V(Q) = \{x, y\}$.

When x and y are non-adjacent vertices of G, we say $S \subseteq V(G) \setminus \{x,y\}$ separates x from y if there is no path from x to y in $G - S$.

Corollary 3.30: Let G be a graph with non-adjacent vertices x and y.
The minimum size of a set that separates x and y is equal to the maximum number of internally disjoint (x,y)-paths.

The neighborhood of a vertex x, denoted $N(x)$, is the set of vertices adjacent to x.

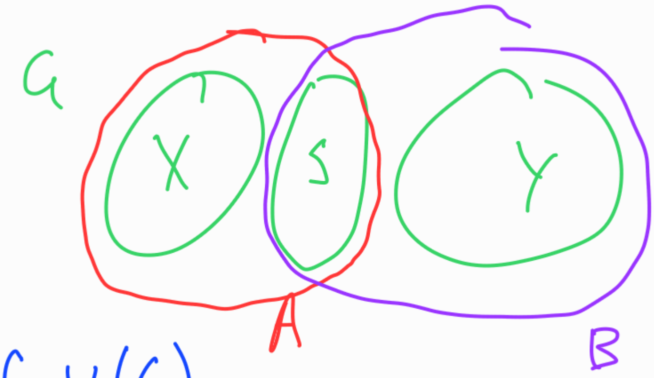


Full details of the proof of Corollary 3.30 in online notes.

Corollary 3.31: Let G be a graph with more than k vertices, for some positive integer k.

G is k-connected if and only if for every pair of vertices u, v in G, there are k internally disjoint (u,v)-paths.

Separations



Let G be a graph and let A and B be subsets of $V(G)$.

We say $\{A, B\}$ is a separation if

i) $A \cup B = V(G)$

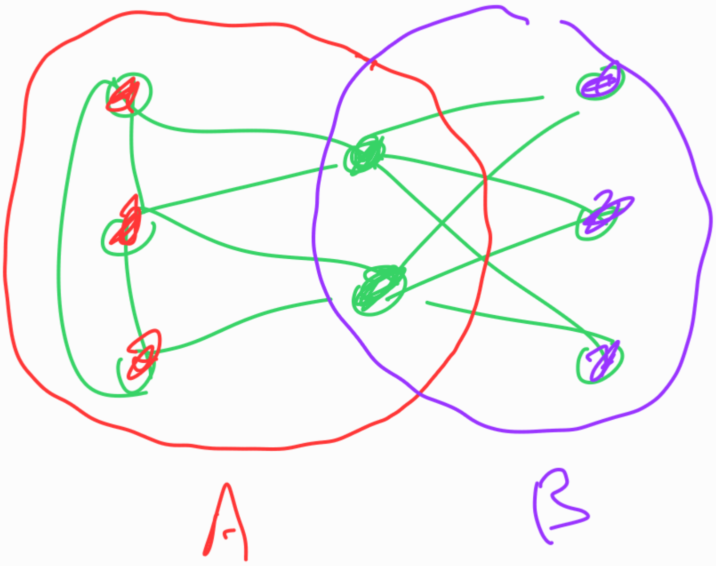
ii) there is no edge of G with one end in $A \setminus B$ and the other in $B \setminus A$.

We call $A \cap B$ the boundary of the separation $\{A, B\}$.

and $|A \cap B|$ is the order of the separation

A separation $\{A, B\}$ is proper if both $A \setminus B$ and $B \setminus A$ are non-empty.

e.g.-



$\{A, B\}$ is a proper separation of order 2.