MATH 361 | Lecture 15

Test: 15 April

Last time: • Whitney's theorem:

Let G be a loopless 2-connected plane graph.

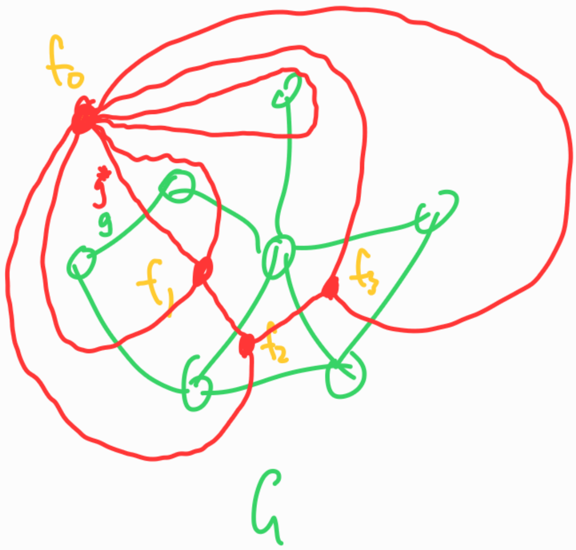
Then for every face f of G, the boundary of f is a cycle.

• If G is planar, then any minor of G is planar.

Today: Planar duals.

Given a plane graph G, we can define another plane graph G^* called the planar dual of G

e.g.



$$V(G^*) = \{f^* : f \in F(G)\}$$

$$E(G^*) = \{e^* : e \in E(G)\}$$

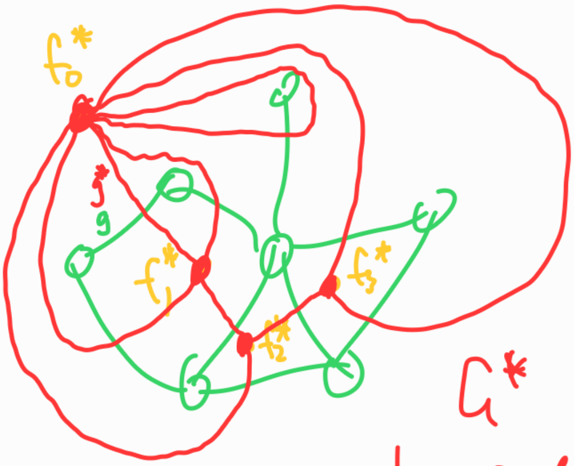
$e^* \in E(G^*)$ has ends f_1^* and f_2^* where $e \in E(G)$ is incident with the faces f_1 and f_2 of G

(where $f_1 = f_2$ if e is a bridge).

In the planar embedding of G^*, each $f^* \in V(G^*)$ is a point in f, and $e^* \in E(G^*)$ with ends f_1^* and f_2^* is a curve from f_1^* and f_2^* that crosses only e, precisely once.

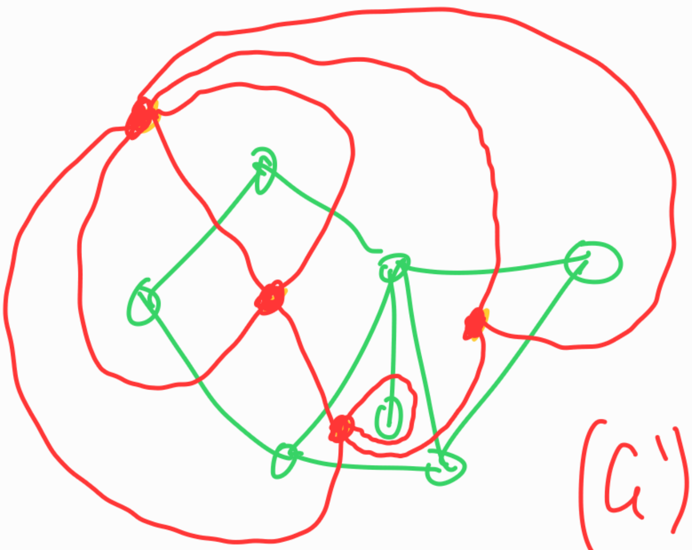
Note: the embedding of the plane graph affects the planar dual.

e.g.



G^* has a vertex with degree 5...

G



$(G')^*$ does not.

G'

Note: for a face f in G

$d(f)$ = [# of edges in a closed walk along the boundary of f]

face degree in G

= [# of edges incident with the vertex f^* in G^*]

$= d(f^*)$

$\uparrow$ vertex degree in G^*

Lemma 4.14 (The Handshaking lemma for faces)

Let G be a plane graph. Then

$$\sum_{f \in F(G)} d(f) = 2|E(G)|.$$

Proof: G has a planar dual G^*.

For each face f of G, there is a vertex f^* of G^* where $d(f) = d(f^*)$.

So $$\sum_{f \in F(G)} d(f) = \sum_{f^* \in V(G^*)} d(f^*)$$

by the Handshaking lemma $$= 2|E(G^*)|$$

$$= 2|E(G)|$$ as required. □

We've seen a natural correspondence between

$E(G)$ and $E(G^*)$

$F(G)$ and $V(G^*)$

What about between

$V(G)$ and $F(G^*)$?

Q

Can you find a plane graph G with a planar dual G^* such that $|V(G)| \neq |F(G^*)|$?

G

has two vertices

G^*

has one face

However, we only have this problem when G is disconnected.

Lemma 4.16: If G is a plane graph, then G^* is connected.

How do we show G^* is connected?

Note that a path $f_1^*, e_1^*, f_2^*, e_2^*, \dots, f_t^*$ in G^* corresponds to a "path" $f_1, e_1, f_2, e_2, \dots, f_t$ in G along faces $f_1, f_2, \dots, f_t$ where e_i is incident to faces f_i and f_{i+1} in G.

So the strategy is: show there is such a "path" between any pair of faces in G.

For a vertex v, let $\partial(v)$ be the set of edges incident with v.

Lemma 4.17 Let G be a connected plane graph

i) If $v \in V(G)$, then $\{e^* : e \in \partial(v)\}$ is the set of edges in the boundary of a face of G^*

ii) If $h \in F(G^*)$, then $\{e^* : e \in \partial(h)\}$ is the set of edges incident with a vertex in G.

Note, we define $(e^*)^* = e$.

Theorem 4.18: If G is a connected plane graph then $(G^*)^* = G$.