MATH361 | Lecture 19

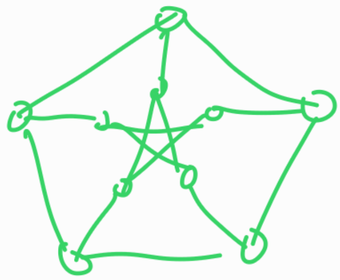
Assignment 3 due 30 April

Recap: Wagner's Theorem:

a graph is planar if and only if it has no K_5-minor 

and no $K_{3,3}$-minor. 

e.g.



the Petersen graph is not planar (see tutorial question).

A class of graphs $\mathcal{G}$ is minor closed if whenever $G \in \mathcal{G}$ and G has the graph H as a minor, then $H \in \mathcal{G}$.

e.g. 1. we seen that planar graphs are a minor-closed class

e.g. 2 the class of forest is a minor-closed class

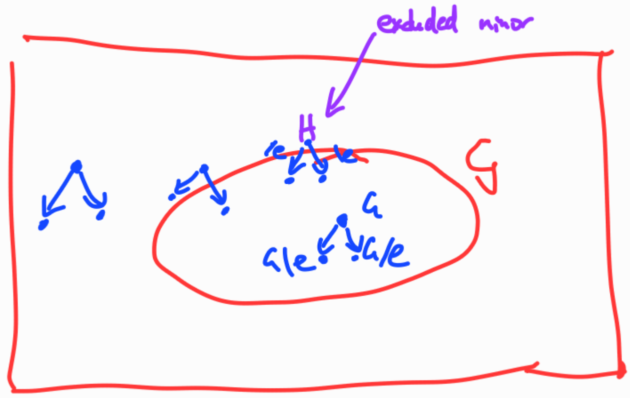
e.g. 3 the class of graphs where every component is a path is a minor-closed class

e.g. 4 for any surface S (e.g. the torus, etc.) the class of graphs embeddable on S is minor-closed

Recall that a minor H of G is proper if $H \not\cong G$.

Let $\mathcal{G}$ be a minor-closed class of graphs.

An excluded minor for $\mathcal{G}$ is a graph H such that $H \notin \mathcal{G}$ but every proper minor of H is in $\mathcal{G}$.



Observation: For a minor-closed class $\mathcal{G}$:

A graph $G \in \mathcal{G}$ if and only if it does not have, as a minor, one of the excluded minors for $\mathcal{G}$.

Proof: $(\Rightarrow)$ If G has H as a minor, where H is an excluded minor for $\mathcal{G}$, then $G \notin \mathcal{G}$ (since $H \notin \mathcal{G}$).

$(\Leftarrow)$ If $G \notin \mathcal{G}$, then either G is an excluded minor, or it has a proper minor that is not in $\mathcal{G}$. Let H is a minimal minor of G not in $\mathcal{G}$. Then H is an excluded minor, and G has H as a minor. $\square$

Takeaway: if we know all the excluded minors for a minor-closed class, that gives us a characterization (like Wagner's theorem).

e.g. 1 Theorem 5.10 rephrased

The excluded minors for the class of planar graphs are K_5 and $K_{3,3}$.

e.g. 2  is the only excluded minor for the class of forests.

<u>Observation</u>: A graph is a forest if and only if it has no -minor.

Any minor-closed class of graphs has a list of excluded minors... but seemingly this could be an infinite list.

<u>Theorem</u> 5.13 (Robertson–Seymour)

Let $\mathcal{G}$ be a minor-closed class of graphs. Then $\mathcal{G}$ has a <u>finite</u> list of excluded minors.

This tells us, for example, that there is a finite list of graphs $\mathcal{T}$ such that

a graph is embeddable on the torus if and only if it has no minor in $\mathcal{T}$.

However, in this case, the complete list $\mathcal{T}$ is not known — what is known is that there are more than 17,535 graphs in $\mathcal{T}$ (Myrvold and Woodcock 2018)

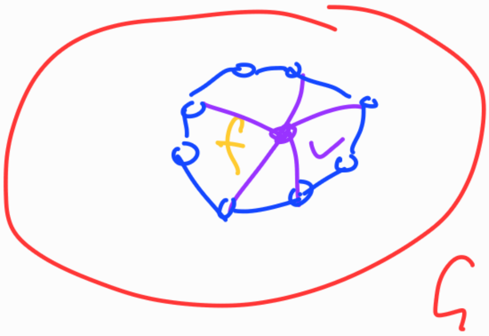
Recall:

the neighborhood of a vertex v, denoted $N(v)$, is the set of all vertices adjacent to v.

Lemma 5.14: Let G be a loopless 3-connected plane graph. For any $v \in V(G)$, there is a cycle in G that contains all the vertices in $N(v)$.

Proof: Let $v \in V(G)$.

$G - v$ is a loopless 2-connected plane graph.



Consider the face f of $G - v$ that contained v. Then the face boundary of f is a cycle (by Whitney's Theorem) and any vertex in the neighborhood of v is in this cycle. $\square$