MATH 361 | Lecture 20

Assignment 3 due Tuesday 3pm

Recap:

Every minor-closed class of graphs can be characterized by a finite set of excluded minors (Robertson-Seymour Theorem)

e.g. $\{K_5, K_{3,3}\}$ is the set of excluded minors for the class of planar graphs (Wagner's Theorem).

Non-example: Recall: a graph is bipartite iff it does not contain an odd cycle (as a subgraph)

The class of bipartite graphs is closed under subgraphs (but not under minors)

When G is an odd cycle, G is not bipartite, but every proper subgraph is bipartite

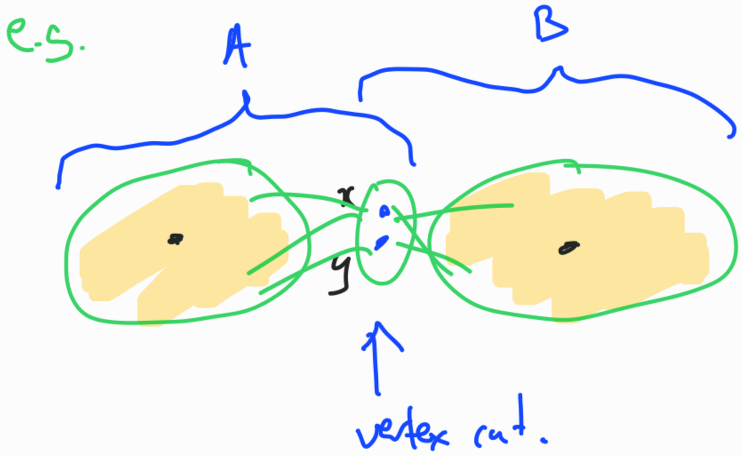
That is, there are infinitely many subgraph-minimal obstructions for bipartite graphs.

In contrast, there are only finitely many minor-minimal obstructions for any minor-closed class of graphs (by R-S Theorem).

Wagner's Theorem A graph is planar if and only if it has no K_5- or $K_{3,3}$-minor.

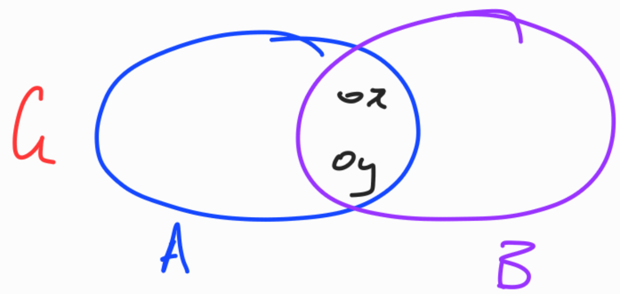
Towards a proof, we recall that for a separation $\{A, B\}$ we have

- $A \cup B = V(G)$
- no edges between $A \setminus B$ and $B \setminus A$



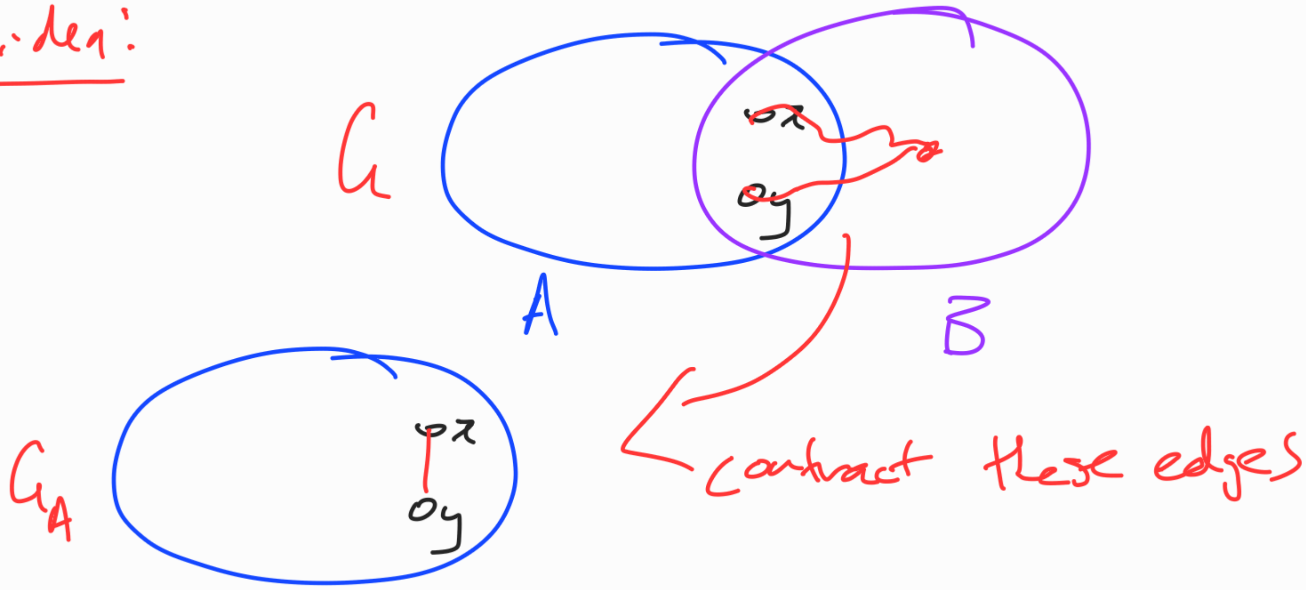
Let G be a 2-connected graph with a proper separation $\{A, B\}$ of order 2. Let $\{x, y\}$ be the vertices in the boundary. We let G_A denote the graph obtained from $G[A]$ by adding an edge between x and y.

Similarly G_B is obtained from $G[B]$ by adding an edge between x and y.



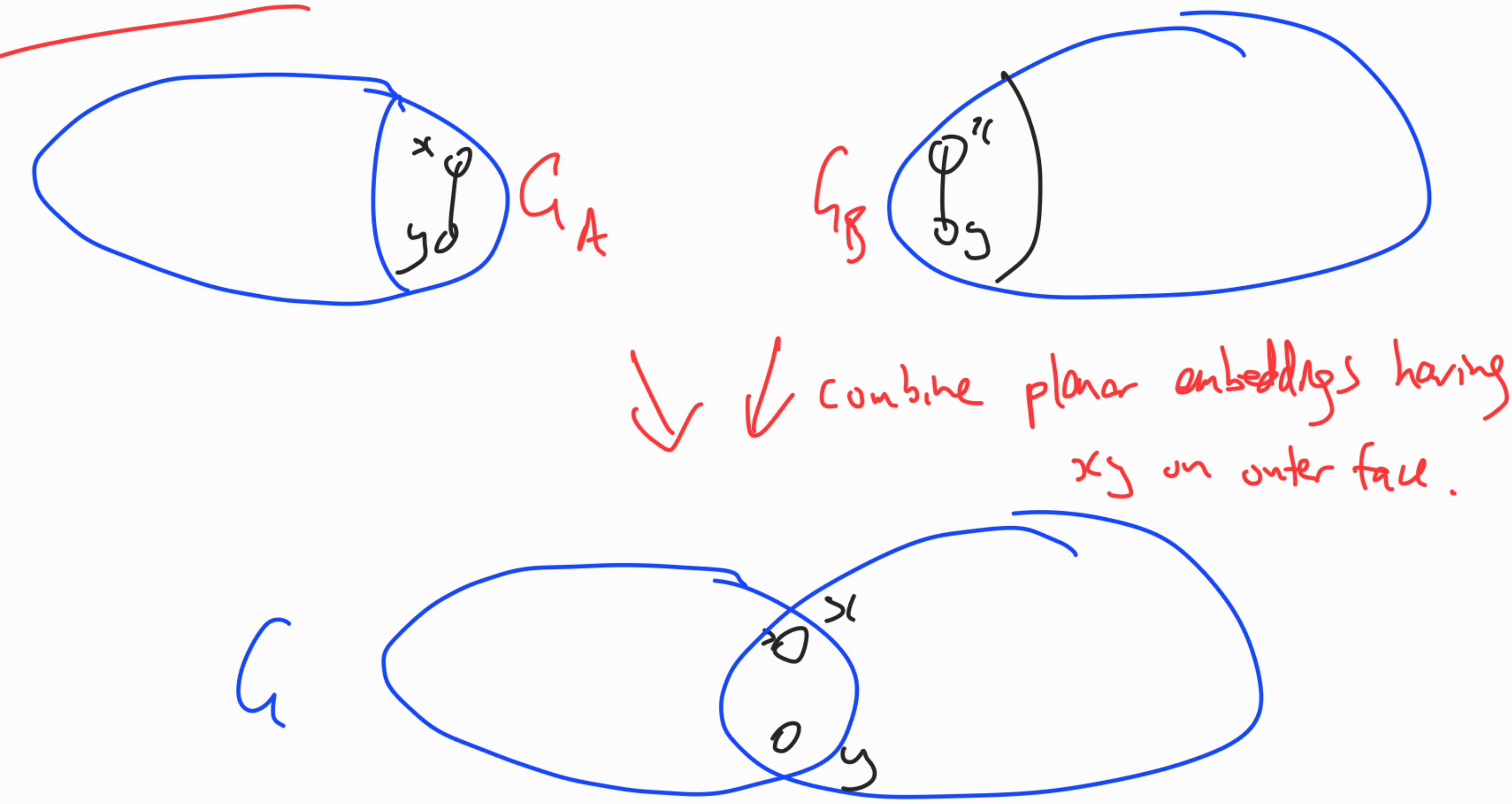
Lemma 5.15: G has G_A and G_B as a minor.

Proof idea:



Lemma 5.16: If G_A and G_B are planar, then G is planar.

Proof idea:



Equivalently, if G is not planar, then
G_A or G_B is not planar.

Proof of Wagner's Thm: (Thm 5.10)

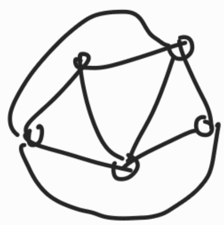
We observed previously that if a graph is planar, then it has no K_5- or $K_{3,3}$- minor. The real task is to show the converse.

Suppose G is not planar. We want to show that G has a K_5- or $K_{3,3}$-minor.

We may assume G is simple.
The proof is by induction on the number of vertices of G.

Base case:

Observe that $K_5 \backslash e$ is planar.



It follows that the only simple graph on at most 5 vertices that is not planar is K_5. Therefore, the result holds for a graph with at most 5 vertices.

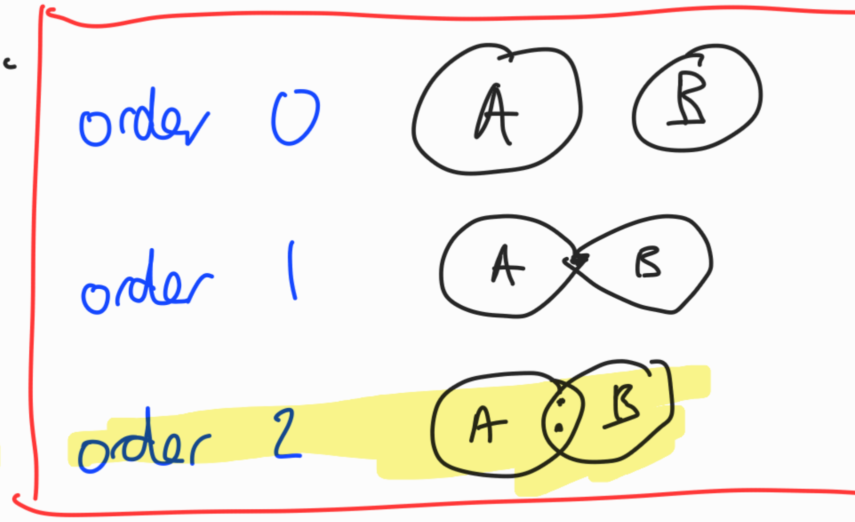
Induction:

Assume that $|V(G)| > 5$ and the result holds for any graph with fewer than $|V(G)|$ vertices.

Claim 1: If G is not 3-connected, then G has a K_5- or $K_{3,3}$-minor.

Say G is not 3-connected. Then G has a proper separation $\{A, B\}$ of order at most 2.

Suppose this separation has order 2.



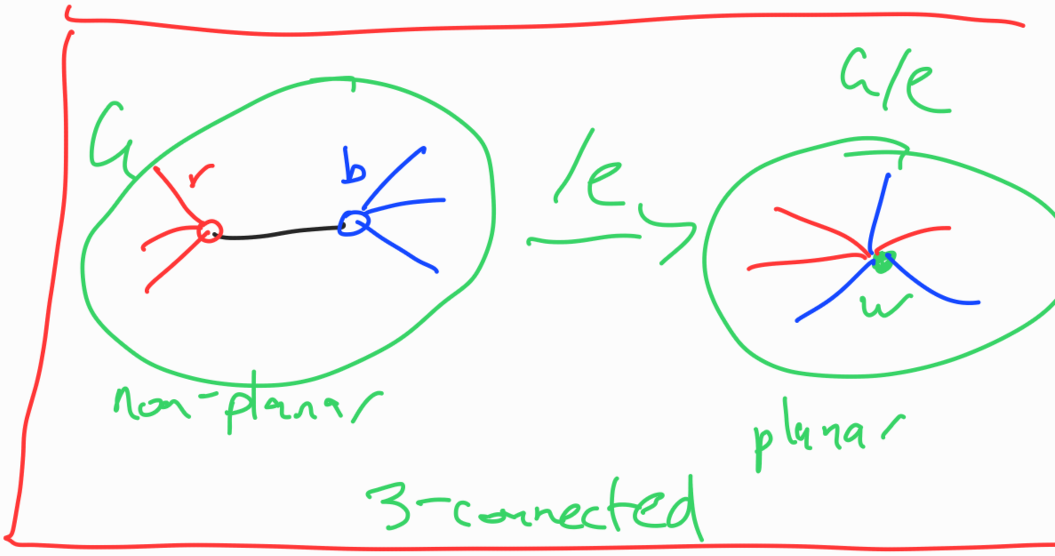
Since G is not planar, by Lemma 5.16, and without loss of generality, G_A is not planar. As G_A has fewer vertices than G, it has a K_5- or $K_{3,3}$-minor (by induction assumption), implying that G has a K_5 or $K_{3,3}$-minor (using Lemma 5.15) as required.

The argument is similar when the separation has order 0 or 1.

It remains to consider when G is 3-connected.

Setup: There exists $e \in E(G)$ such that G/e is 3-connected (by Thm 3.12).

If G/e is not planar, then by the induction assumption it has a K_5- or $K_{3,3}$-minor, and hence so does G. So we may assume G/e is planar.



Let $e = rb$ and let w be the vertex resulting from the contraction of e.

By Lemma 5.14, the neighbours of w are in a cycle C of G/e.

Let R be the edges incident with r in G
B b in G

Say a vertex of C is red if it is incident with an edge in R
blue B
coloured if it is red or blue (or both)

There is a natural cyclic ordering

$(v_1, v_2, \ldots, v_k)$ on the colored vertices

of C using the planar embedding of G/e.

To finish, we consider what colorings of our

cyclic ordering allow for planar/non-planar

graphs.