MATH361 | Lecture 21

Assignment 3 due 3pm tomorrow

Last time: partway through a proof of Wagner's theorem:

A graph is planar iff it has no K_5- or $K_{3,3}$-minor.

Proof: Let G be a non-planar graph. We want to show G has a K_5- or $K_{3,3}$-minor.

Proof by induction on $|V(G)|$.

Induction assumption: Assume that $|V(G)| > 5$ and that the result holds for any graph with fewer vertices than G.

Claim 1: If G is not 3-connected then it has a K_5- or $K_{3,3}$-minor.

So we may assume G is 3-connected.

Setup: there exists $e \in E(G)$ such that G/e is 3-connected

We may assume G/e is planar.

Let $e = rb$ and let w be the vertex resulting from the contraction of e.

By Lemma 5.14, the neighbors of w are in a cycle C of G/e.

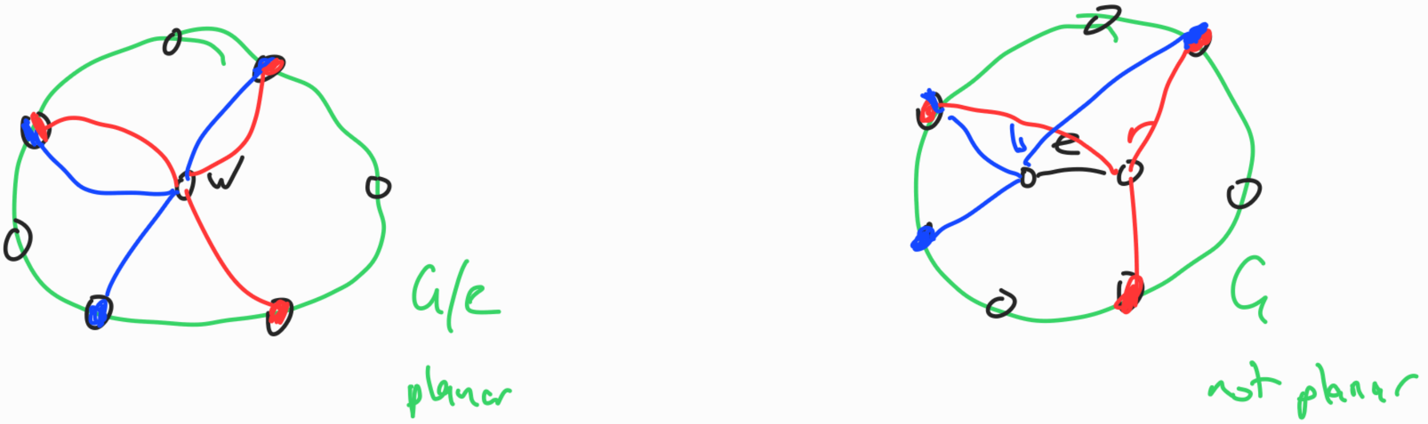
A vertex in C is red if it is adjacent to r in G

blue if it is adjacent to b in G

coloured if it is red, blue or both.

There is a natural cyclic ordering $(v_1, v_2, \dots, v_t)$ on the coloured vertices of C from the planar embedding of G/e.

e.g.



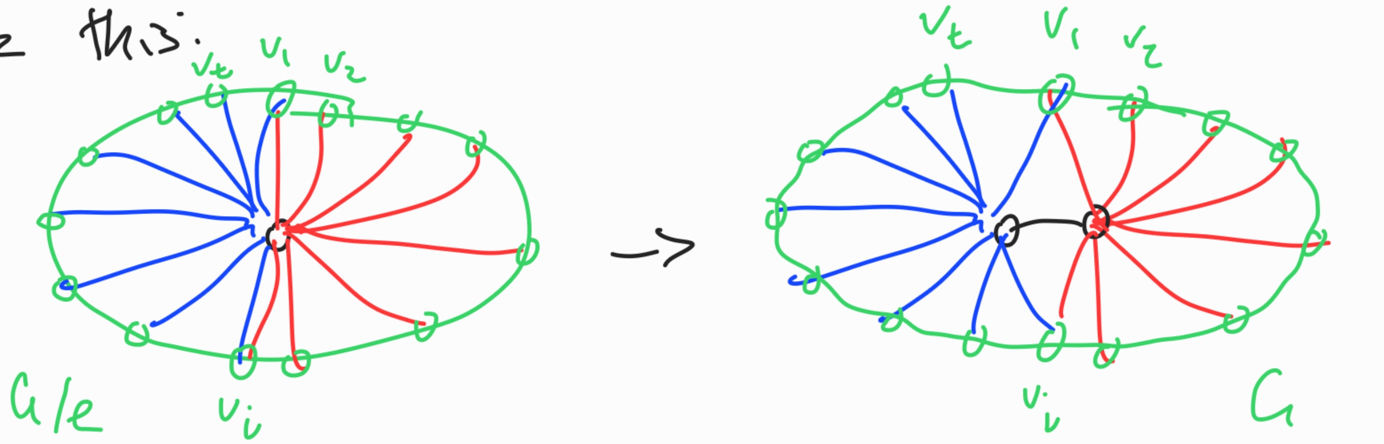
Claim 2: If we have a cyclic ordering

$$(v_1, \underbrace{v_2, \dots, v_{i-1}}_{\text{red and not blue}}, v_i, \underbrace{v_{i+1}, \dots, v_t}_{\text{blue and not red}}),$$

anything (red/blue/both)

then G is planar.

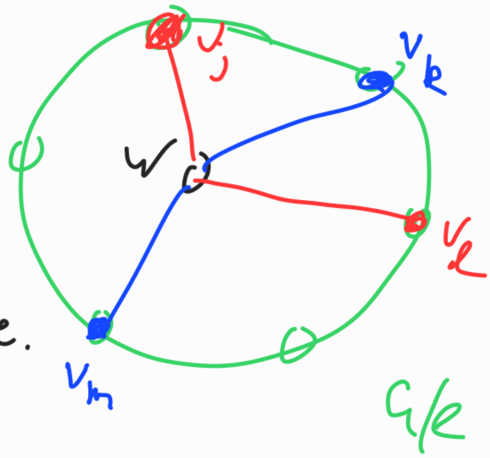
To see this:



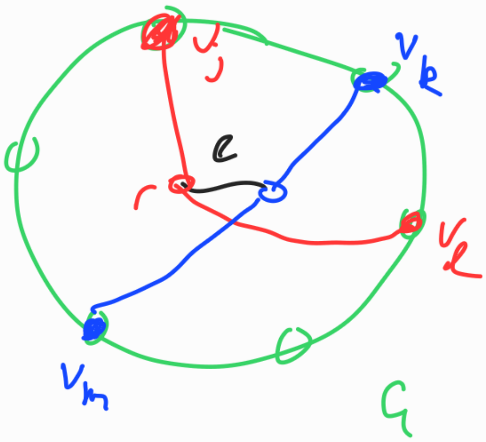
Claim 3: If C has at least 4 coloured vertices, then G has a $K_{3,3}$-minor.

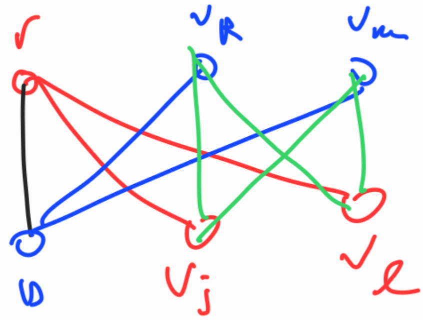
Assume C has at least 4 colored vertices.

Then we must have a cyclic ordering $(v_1, \dots, v_t)$ such that $j < k < \ell < m$, v_j and v_ℓ are red, and v_k and v_m are blue.



Starting from G and deleting all edges other than those illustrated, and contracting edges of C that are incident to a non-coloured vertex, we obtain the following $K_{3,3}$-minor:





This proves Claim 3.

<u>Claim 4</u>: If C has exactly 3 colored vertices, then G has a K_5-minor.

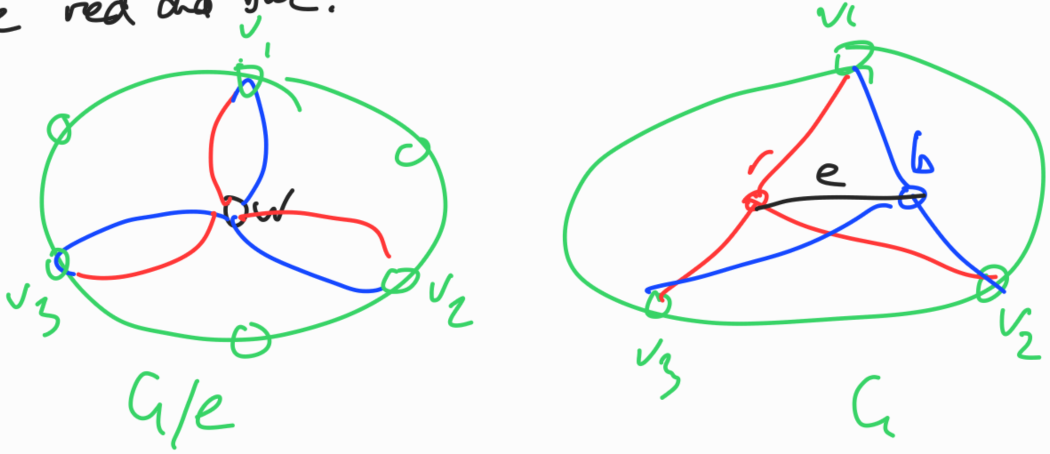
Assume we have precisely 3 colored vertices, so (v_1, v_2, v_3) is the cyclic ordering of the colored vertices of C.

If v_3 is blue and not red, then we can apply Claim 2.

The same is true if any vertex is blue and not red or red and not blue.

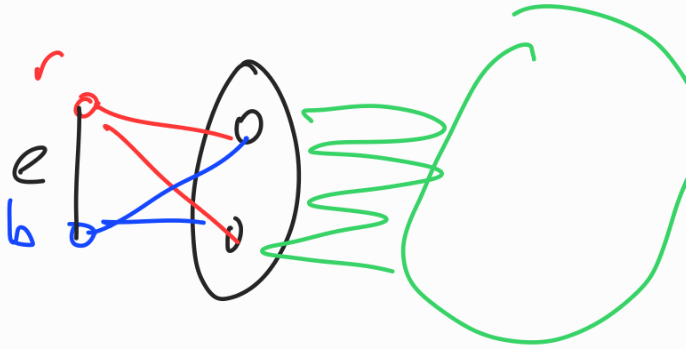
So v_1, v_2, and v_3 are red and blue.

Then we have:



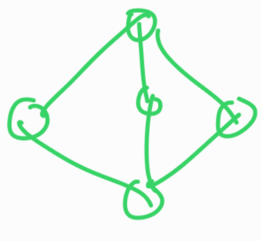
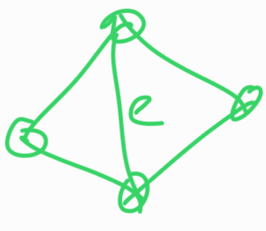
So, using the same approach as in the previous claim, we obtain a K_5 minor, as illustrated on the right. This proves Claim 4.

If C has at most 2 coloured vertices, they form a vertex cut, so G is not 3-connected.



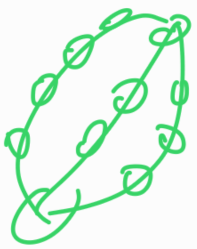
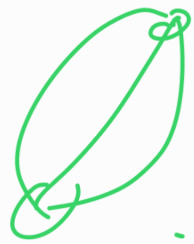
The result now follows from Claims 1-4 and induction. □

An edge subdivision:

e.g.  is obtained from 

by subdividing e.

A graph H is a <u>subdivision</u> of a graph G if H can be obtained from G by a sequence of (zero or more) edge subdivisions.

e.g.  is a subdivision of  .

<u>Theorem 5.16</u> (Kuratowski's Theorem; 1930)

A graph is planar if and only if it does not contain a subgraph that is isomorphic to a subdivision of K_5 or $K_{3,3}$.