MATH361 | Lecture 23

Assignment 4 due 14 May

Recap:

Wagner's Th^{n}: A graph is planar iff it has no minor isomorphic to K_5 or $K_{3,3}$.

Kuratowski Th^{n} A graph is planar iff it has no topological minor isomorphic to K_5 or $K_{3,3}$.

$\boxed{\text{H is a topological minor of G if G has a subgraph that is a subdivision of H}}$

H is a topological minor of G $\Rightarrow$ H is a minor of G
(Lemma 5.17)

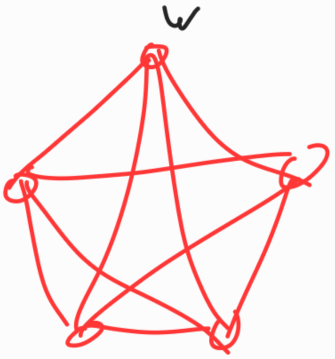
H is a topological minor of G $\Leftarrow$ H is a minor of G and H is cubic.
(Thm 5.19)

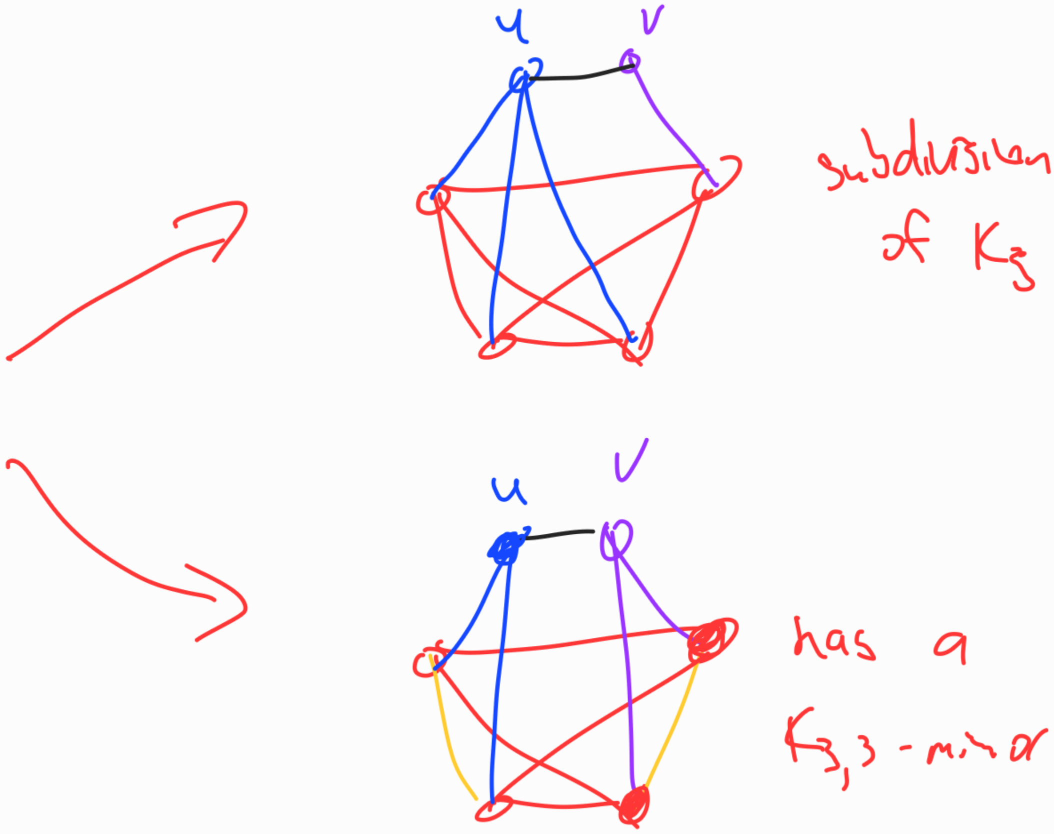
$K_{3,3}$ is a topological minor of G $\Leftarrow$ $K_{3,3}$ is a minor of G
(Corollary 5.20)

Corollary 5.21 Let G be a graph with a K_5-minor.

Then either K_5 or $K_{3,3}$ is a topological minor of G.

Key idea:



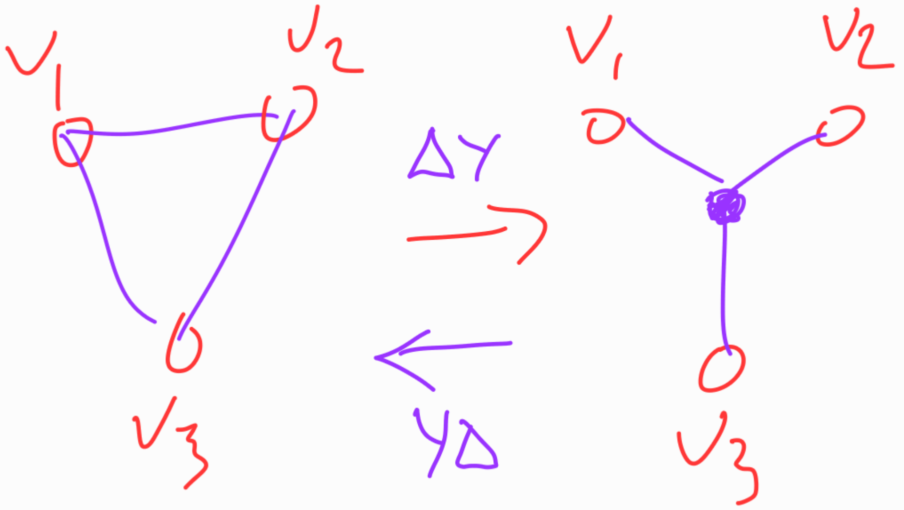


<u>Proof</u> of Kuratowski's theorem:

Suppose G is not planar. Then G has a K_5- or $K_{3,3}$-minor by Wagner's theorem. By Corollaries 5.20 + 5.21, G has K_5 or $K_{3,3}$ as a topological minor.

Now suppose G has K_5 or $K_{3,3}$ as a topological minor. Then, by Lemma 5.17, G has a K_5 or $K_{3,3}$-minor. Thus, by Wagner's Theorem, G is not planar. $\square$

ΔY - exchange

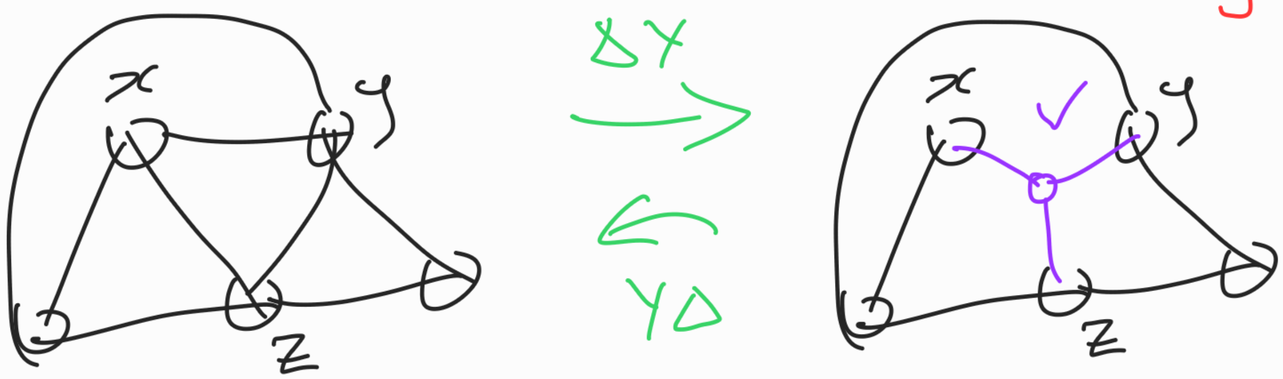


A triangle in a graph is a cycle of length 3

A triad in a graph is a subgraph induced by the edges incident with a degree-3 vertex with 3 distinct neighbors

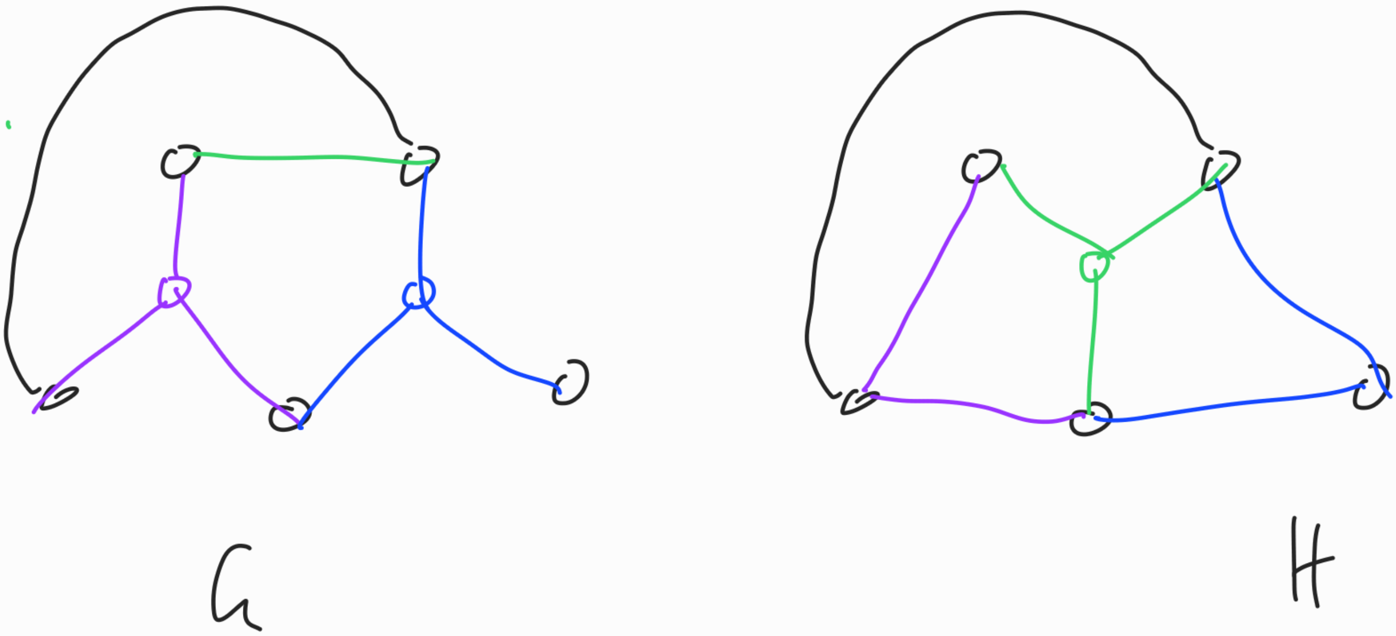
Intuitively a ΔY exchange changes a triangle into a triad while a YΔ exchange changes a triad into a triangle.

e.g.



Two graphs G and H are ΔY-equivalent if there is a sequence of ΔY- and $Y\Delta$ exchanges that transform G into H.

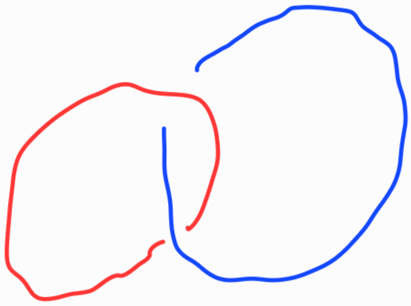
e.g.



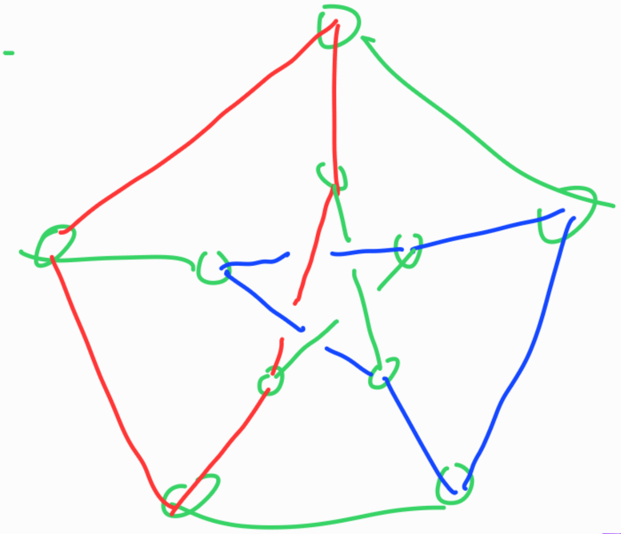
G and H are ΔY-equivalent.

Consider graphs that we can embed in $\mathbb{R}^3$ but with a particular property:

no two cycles are "linked"

(think: )

e.g.



An embedding of the Petersen graph in $\mathbb{R}^3$

The red and blue cycles are linked.

It turns out that any embedding of the Petersen graph in $\mathbb{R}^3$ will have some pair of linked cycles

A graph that can be embedded in $\mathbb{R}^3$ with no two linked cycles is linklessly embeddable

The Petersen graph is not linklessly embeddable.

The class of linklessly embeddable graphs is minor closed

The Petersen family is the set of seven graphs that are ΔY-equivalent to the Petersen graph (one of which K_6)

In 1992, Robertson, Seymour and Thomas proved:

Theorem 5.23: A graph is linklessly embeddable if and only if it has no member of the Petersen family as a minor

One more interesting class:

$Y\Delta Y$-reducible graphs

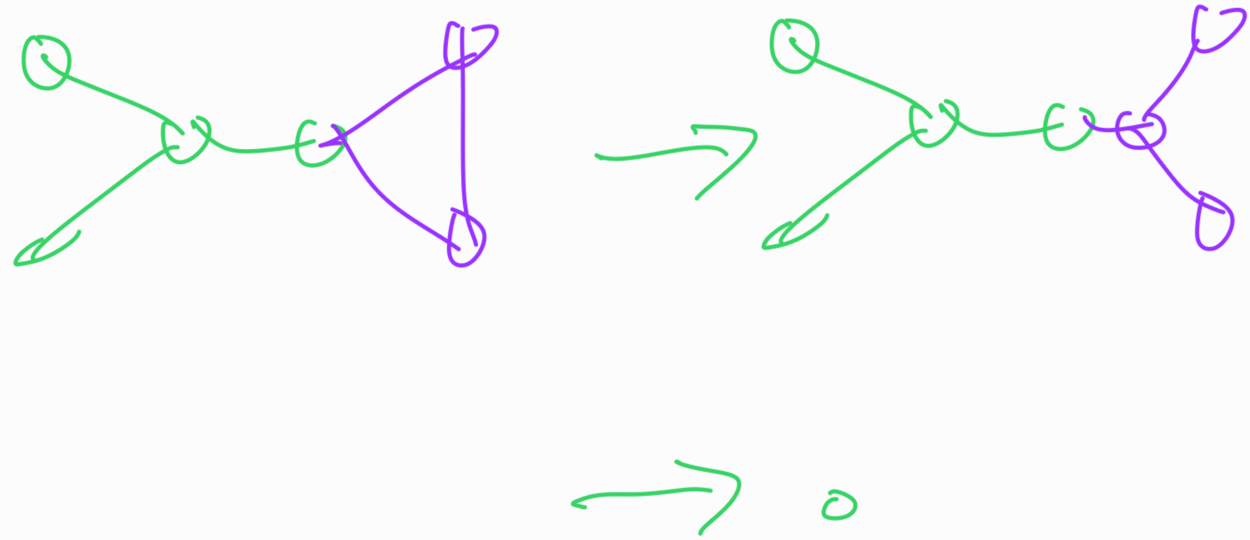
A series-parallel reduction is any one of the following operations:

- deleting a loop
- contracting a bridge
- deleting an edge in a parallel pair
- contracting an edge incident to a degree 2-vertex, that's not in parallel

For graphs G and H, we say that G is $Y\Delta Y$-reducible to H if H can be obtained from G by a sequence of ΔY exchanges, $Y\Delta$ exchanges and series parallel reductions.

G is $Y\Delta Y$-reducible if it is $Y\Delta Y$-reducible to an edgeless graph

e.g.



G

G is $Y\Delta Y$-reducible.

The Y∆Y-reducible graphs are minor-closed,

However, there are known to be more than 68 million excluded minors for this class (Yi, 2006).