MATH361 Lecture 24

Assignment 4 due 14 May

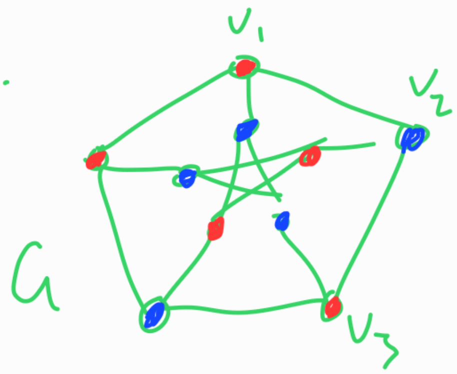
Graph coloring

Let G be a graph. A vertex coloring of G is a function

$$\varphi : V(G) \to S \qquad \text{where } S \text{ is a set.}$$

It is a k-coloring if $|S| = k$.

e.g.



Formally, $\varphi : V(G) \to S$

where $S = \{r, b\}$

$\varphi(v_1) = r$, $\varphi(v_2) = b$, $\varphi(v_3) = r$, etc.

φ is a 2-coloring.

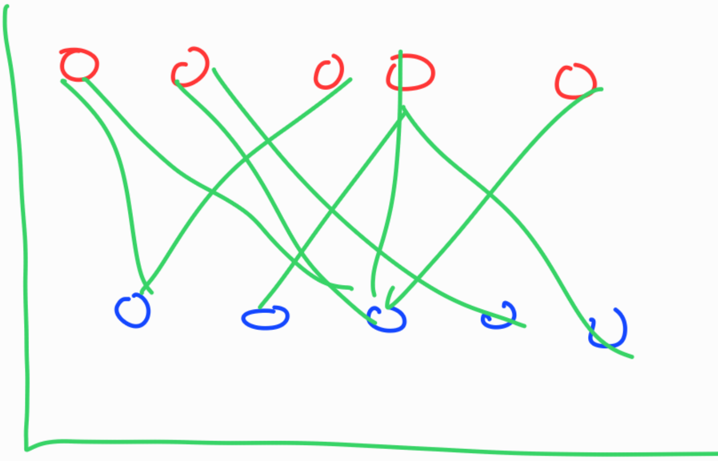
If instead $S = \{r, b, g\}$, then φ is a 3-coloring.

A coloring is proper if for every edge $e = uv$ we have $\varphi(u) \neq \varphi(v)$.

(Here $u = v$ when e is a loop).

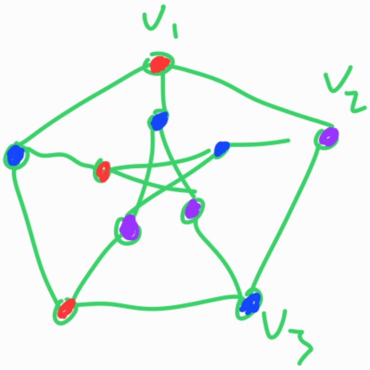
A graph is k-colorable if it has a proper k-colouring.

e.g. 1 For a graph G, the following are equivalent:



- G is bipartite
- G is 2-colourable
- G has no odd cycles (see Theorem 1.7 in course notes)

e.g. 2 The earlier 2-colouring of the Petersen graph was not proper. It is not 2-colourable, since it contains an odd cycle, however it is 3-colourable, as illustrated.



Determining if a graph is 2-colourable is computationally easy (polynomial-time solvable) whereas determining if a graph is 3-colourable is hard (NP-complete).

Observation: Let j and k be positive integers. If a graph is k-colorable, then it is j-colorable for all $j \geq k$.

The chromatic number of a graph G is the smallest positive integer k such that G is k-colourable, or ∞ if no such number exists.

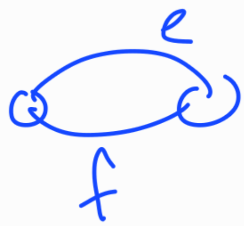
e.g. 1 A bipartite graph with at least one edge has chromatic number 2.

eg. 2 The Petersen graph has chromatic number 3.

e.g. 3. The complete graph on n vertices has chromatic number n, since any 2 vertices must have different colors.

e.g. 4 A graph with a loop has chromatic number ∞

A graph with a parallel pair $\{e, f\}$ is k-colorable iff $G \setminus e$ is k-colourable.



So we usually focus on coloring simple graphs.

For a graph G, the maximum degree of G is

$$\max_{v \in V(G)} d(v)$$

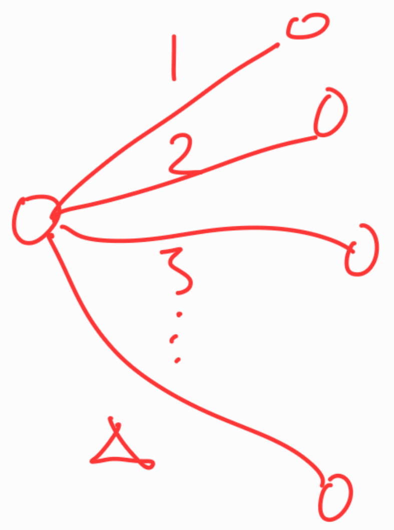
e.g. the maximum degree of $\begin{cases} K_n \text{ is } n-1 \\ K_{1,m} \text{ is } m \end{cases}$

Lemma 6.7 Let G be a simple non-empty

graph with maximum degree Δ. Then G is $(\Delta+1)$-colourable.

<u>Proof:</u> Proof by induction on the number of vertizes.

Idea:



at most Δ neighbours but $\Delta+1$ color options.

If G has exactly 1 vertex v, then $d(v) = 0$ and G is 1-colourable, so the result holds in this case.

Assume $|V(G)| \geq 2$, and the result holds for a graph with $|V(G)| - 1$ vertices.

Pick an arbitrary vertex v in G. By the induction assumption $G-v$ is $(\Delta'+1)$-colourable, where Δ' is the maximum degree of $G-v$.

Note that $\Delta' \leq \Delta$, so $G-v$ is $(\Delta+1)$-colourable.

Let $\varphi' : V(G-v) \to S$ be a proper $(\Delta+1)$-colouring of $G-v$ on the colours $S = \{1, 2, \dots, \Delta+1\}$.

Since $d(v) \leq \Delta$, we have $|N(v)| \leq \Delta < |S|$.

We extend φ' to a proper $(\Delta+1)$-colouring φ of G by finding a colour for v not used by any of its neighbours. That is let $c \in S \setminus \{\varphi'(u) : u \in N(v)\}$

Then
$$\varphi(u) = \begin{cases} \varphi'(u) & \text{when } u \in V(G-v) \\ c & \text{when } u = v \end{cases}$$

is a $(\Delta+1)$-coloring of G. The result follows by induction. $\square$

Lemma 6.7 gives us an upper bound on the chromatic number of a graph.

e.g. K_n has maximum degree $n-1$ and chromatic number n.

$K_{m,m}$ has maximum degree m

but has chromatic number 2.