MATH361 | Lecture 25

Assignment 4 due next Tuesday

Recall we are (for now) focussed on proper vertex colourings of simple graphs

A graph is k-colourable if it has a proper (vertex) k-colouring.

A graph is 2-colourable iff it is bipartite

Let Z be a set and let k be a positive integer.

We say $(X_1, X_2, \dots, X_k)$ is a k-partition if

$Z = X_1 \cup X_2 \cup \dots \cup X_k$ and $X_i \cap X_j = \emptyset$ for all distinct $i, j \in \{1, 2, \dots, k\}$.

A graph G is k-partite if there is a k-partition $(V_1, V_2, \dots, V_k)$ of $V(G)$ such that $G[V_i]$ has no edges for $i \in \{1, 2, \dots, k\}$.

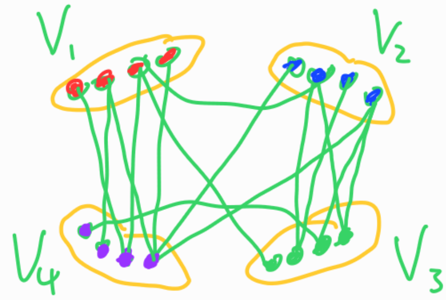
A set of vertices $X \subseteq V(G)$ for which $G[X]$ has no edges is called a stable set.

For a k-partition $(V_1, V_2, \dots, V_k)$, each V_i is a stable set.

Note "bipartite" is the same as "2-partite".

Lemma 6.1: A graph G is k-colourable iff G is k-partite.

e.g.



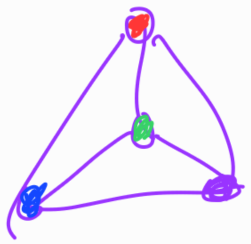
Recall: the chromatic number of a graph G is the smallest positive integer k such that G is k-colorable. It is often denoted $\chi(G)$

Lemma 6.7 A simple non-empty graph with maximum degree Δ is $(\Delta+1)$-colourable.

In other words $\chi(G) \leq \Delta(G)+1$ where $\Delta(G)$ is the maximum degree of G.

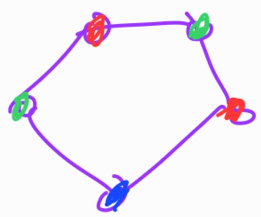
This tells us, for example, that all simple $\left\{ \begin{matrix} \text{cubic} \\ \text{subcubic} \end{matrix} \right\}$ graphs are 4-colourable.

Are there any (simple) connected cubic graphs that are not 3-colorable?



Only K_4

Are there any (simple) connected non-complete graphs G that are not $\Delta(G)$-colorable?



Only odd cycles of length at least 5.

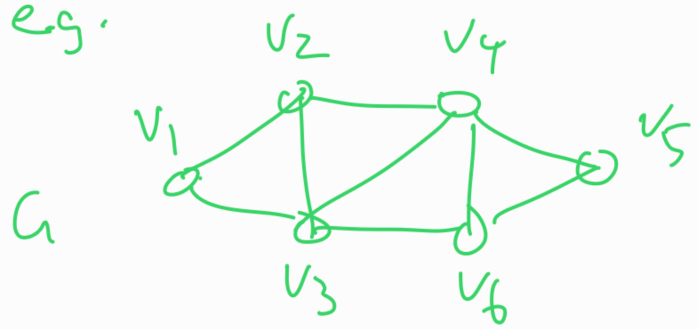
Theorem 6.8 (Brooks' Theorem)

Let G be a simple connected graph with maximum degree Δ. If G is not an odd cycle or a complete graph, then G is Δ-colourable.

Let G be a connected graph, with $v \in V(G)$.

A search ordering of G is an ordering $(v_1, v_2, \dots, v_n)$ of $V(G)$ such that for each i with $2 \le i \le n$, the vertex v_i has a neighbour in $\{v_1, v_2, \dots, v_{i-1}\}$.

e.g.



$(v_1, v_2, v_3, v_4, v_5, v_6)$ is a search ordering of G starting at v_1

For any connected graph G and any $v \in V(G)$, there is a search ordering starting at v.

Proof of Brooks Theorem

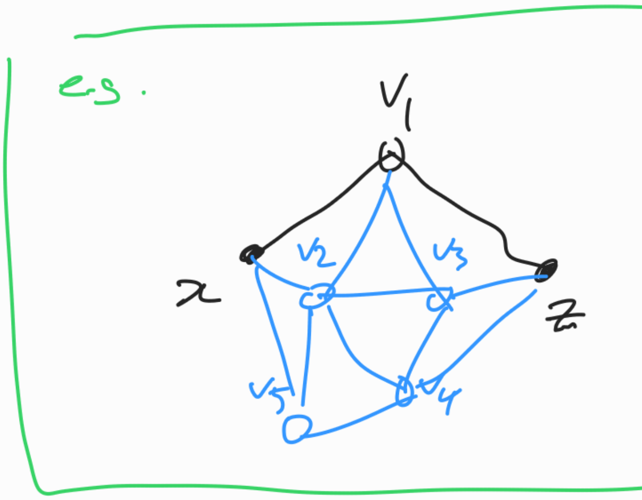
Assume G is not an odd cycle or a complete graph.

Our aim is to show that G is Δ-colorable.

For $x, z \in V(G)$, let $d(x,z)$ denote the length of a shortest (x,z)-path (the "distance" between x and z)

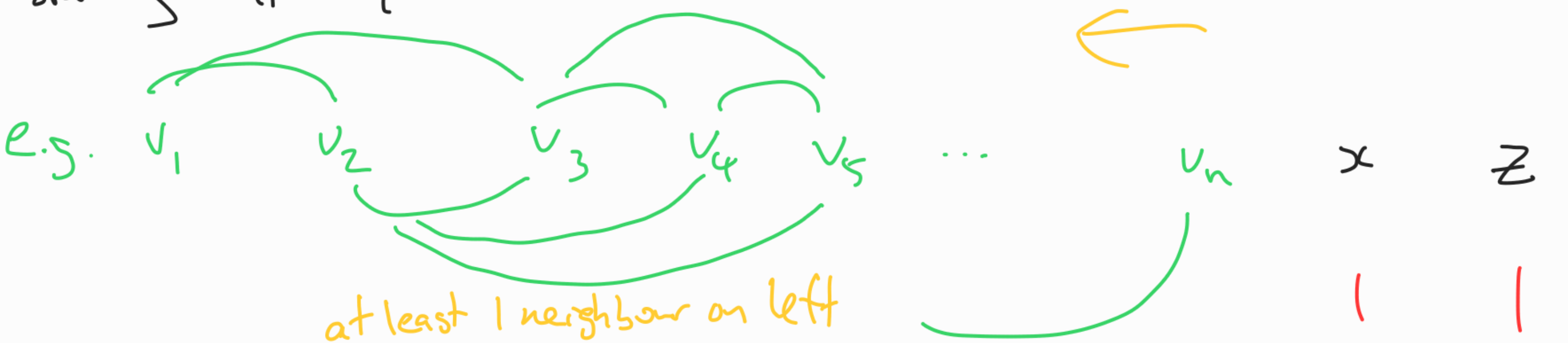
Claim: Suppose G has $x, z \in V(G)$ with $d(x,z)=2$ and $G-\{x,z\}$ is connected. Then G is Δ-colourable.

Proof of claim:



Since $d(x,z)=2$, there is a vertex, v_1 say, that is adjacent to x and z.

Let $(v_1, v_2, v_3, \dots, v_n)$ be a search ordering of $G-\{x,z\}$ starting at v_1.



at least 1 neighbor on left
at most Δ-1 neighbors on right.

We will find a proper Δ-colouring $\varphi : V(G) \to \{1, 2, \dots, \Delta\}$.

Let $\varphi(x)=1$ and $\varphi(z)=1$. We'll colour v_n, then v_{n-1}, then v_{n-2}, ..., down to v_2 as follows: each such v_i has at most $\Delta-1$ neighbors that have already been assigned colours (those "on the right"). (Since we have a search ordering). Thus, we can pick a colour for v_i that is not used by any of its neighbors in $\{x,z\} \cup \{v_j : i < j \leq n\}$. In this way, we obtain a proper Δ-colouring for $G-v_1$. But v_1 has at most Δ

neighbours, two of which are coloured 1. So there is a colour in $\{1, 2, \dots, \Delta\}$ not used by any neighbours of v_1. We set $\varphi(v_1)$ to be this colour, thereby obtaining a proper Δ-coloring of G. This proves the claim.

It remains to show that we can find vertices x and z as described in the claim.

Consider when G is 3-connected

then when G is not 3-connected.