MATH 361 Lecture 26

Assignment 4 due Tues 3pm

Recap:

Every graph G is $(\Delta(G)+1)$-colorable

where $\Delta(G)$ is the maximum degree of G.

Moreover, if G is not a complete graph nor an odd cycle,

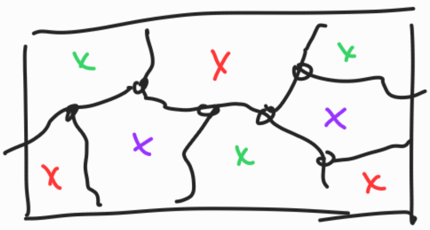
then G is $\Delta(G)$-colourable (Brooks' Theorem).

Brooks Theorem tells us that graphs with only "low" vertex degrees have "low" chromatic number.

But Brooks Theorem only tells us $K_{100,100}$ is 100-colourable — in fact it is 2-colourable.

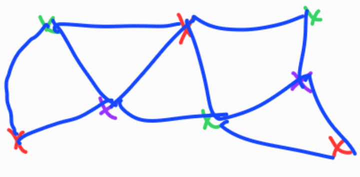
Planarity is another property that ensures low chromatic number.

The 4-colour Theorem



Conjecture (Original formulation)

For any plane graph G with no bridges, we can color the faces of G with one of 4 colors so that adjacent faces get different colors



By planar duality, the following vertex colouring formulation is equivalent:

Conjecture 6.11 (The 4-color conjecture):

Every loopless planar graph is 4-colorable.

Around 1880, Tait and Kempe published "proofs"

1891 Petersen found a flaw — 1890 Heawood found a flaw

1946 Tutte found serious flaw

Appel and Haken published a controversial proof in 1976

2 parts: (1) a large analysis to reduce to a finite case check

(2) case check performed by computer

In 1993 Robertson, Sanders, Seymour, Thomas

came up with a simpler proof using similar ideas

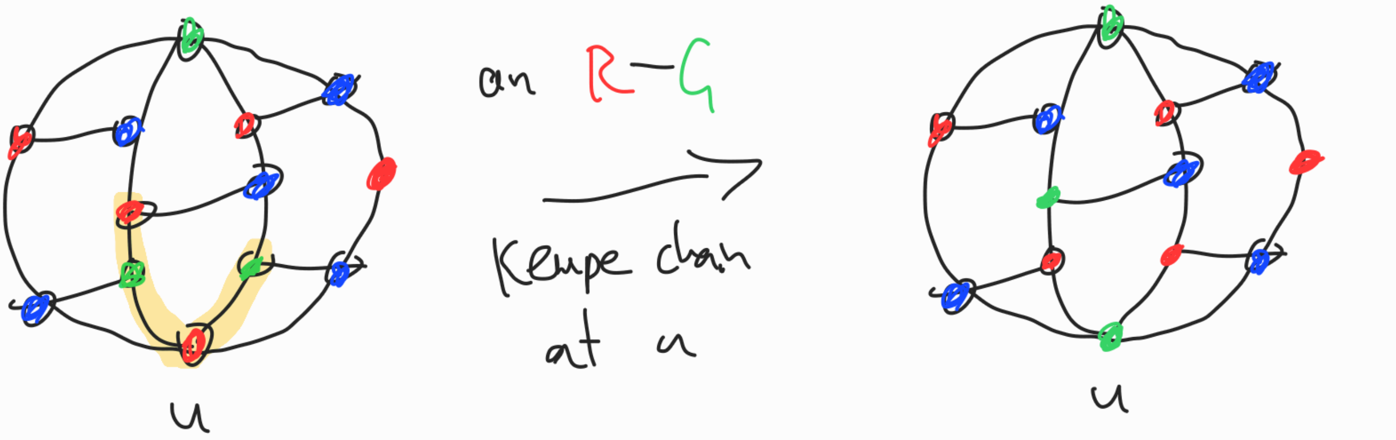
Theorem 5.12 (Heawood + Kempe ~1900)

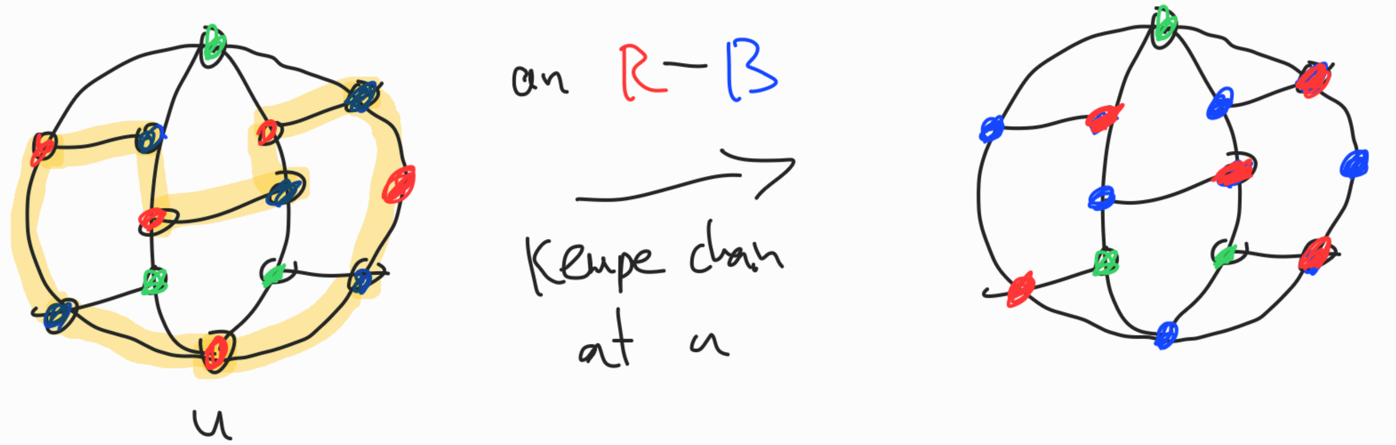
Every loopless planar graph is 5-colorable.

Kempe chains

For an X-Y Kempe chain at u, where u is colored in X, take a

maximum connected subgraph containing u consisting of vertices colored only X or Y, and swap all colours X and Y in this subgraph.





Lemma 6.13: Let G be a graph with a proper k-colouring φ. Let $u \in V(G)$ such that $\varphi(u) = R$. Then φ' be obtained from φ by an R–B Kempe chain at u. Then

i) φ' is a proper k-colouring, and

ii) $\varphi'(v) \neq \varphi(v)$ for $v \in V(G)$ if and only if there is an alternating R–B path from u to v.

Lemma 6.14: Every simple planar graph has a vertex with degree at most 5

Theorem 5.12

Every loopless planar graph is 5-colourable.

Proof: Let G be a loopless planar graph.

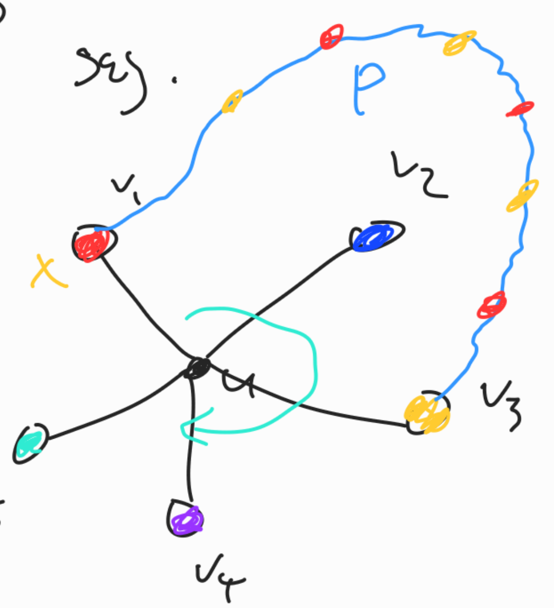
Proof by induction on the number of vertices of G.

The result clearly holds if $|V(G)| \leq 1$.

Assume $|V(G)| \geq 2$ and every loopless planar graph with fewer than $|V(G)|$ vertices is 5-colourable.

By Lemma 6.14, G has a vertex u of degree at most 5.

By the induction assumption, $G-u$ is 5-colourable using the colours $\{R, B, Y, M, C\}$ say.



If u does not have 5 neighbors of 5 different colors, we can extend the colouring of $G-u$ to a colouring of G. So we may assume $d(u)=5$ and the neighbors of u have all different colours.

Consider a planar embedding of G. Suppose the neighbors of u are v_1, v_2, v_3, v_4, v_5 appearing in this cyclic order

We recolor $G-u$ using an R-Y Kempe chain at v_1.

Then v_1 is coloured Y and either

i) v_3 is still colored Y, in which case we colour u R, thereby obtaining a proper 5-colouring of G

ii) v_3 is now colored R, in which, by Lemma 6.13(ii) there is (v_1, v_3)-path P where colours alternate between R and Y

Then consider a B-M Kempe chain at v_2.