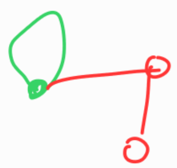
MATH361 Lecture 3

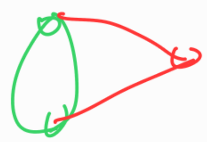
Recap:
- subgraphs
  - induced subgraphs
  - edge-induced subgraphs
- components, connected / disconnected graphs
- cycle

The **length** of a cycle C is the number of edges in C

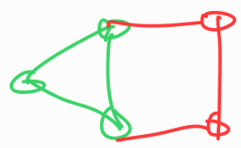
e.g.



length 1



length 2



length 3

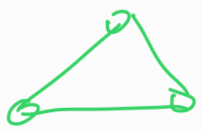
The **length** of a path P is the number of edges in P.

We've seen the **path graphs** e.g. P_4

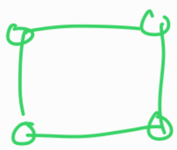
Note: the path graph P_n has a path of length $n-1$

Cycle graphs e.g.

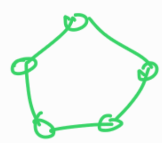
C_3



C_4



C_5



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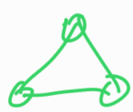
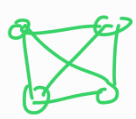
Complete graphs

K_2

K_3



K_4



K_5

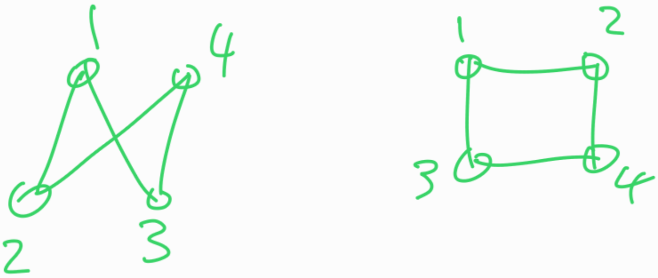
...

a simple graph on n vertices with all possible edges.

Bipartite graphs

A graph G is **bipartite** if the vertices can be partitioned into two sets A and B such that every edge of G has one end in A and the other in B.

e.g



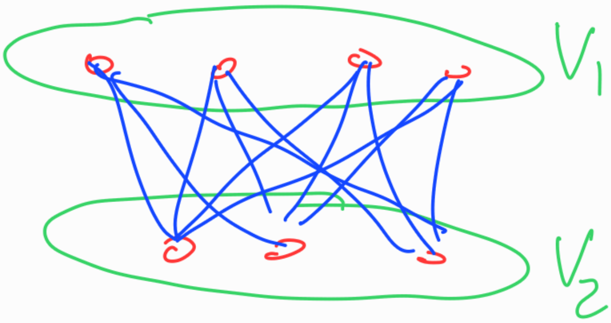
C_4 is bipartite.

The complete bipartite graph $K_{s,t}$ has vertex set $V_1 \cup V_2$ where $|V_1| = s$ and $|V_2| = t$, and V_1 and V_2 are disjoint, and for each $v_1 \in V_1$ and $v_2 \in V_2$ there is a single edge with ends v_1 and v_2.

$|V_1| = s$

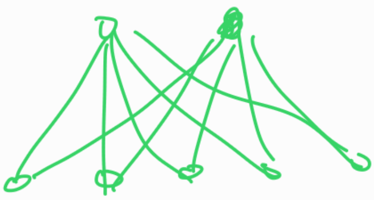
$K_{s,t}$

$|V_2| = t$

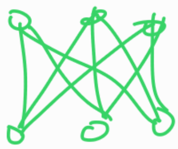


$K_{3,4}$

e.g. $K_{2,5}$



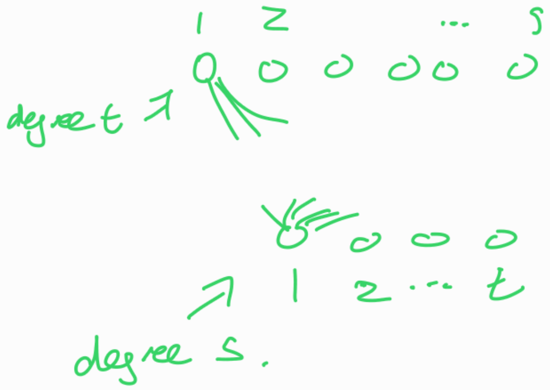
$K_{3,3}$



Aside: $K_{2,5}$ is planar but $K_{3,3}$ is not (see Kuratowski's theorem later!)

Theorem 1.8: For non-negative integers s and t, $K_{s,t}$ has st edges.

Proof: $K_{s,t}$ has s vertices of degree t and t vertices of degree s.



By the Handshaking lemma,

$$2|E(K_{s,t})| = st + ts = 2st$$

so $|E(K_{s,t})| = st$. $\square$

Isomorphism

Let G_1 and G_2 be graphs.

An isomorphism from G_1 to G_2 is a pair of bijections $f: V(G_1) \to V(G_2)$ and $g: E(G_1) \to E(G_2)$ such that $v \in V(G_1)$ is incident with $e \in E(G_1)$ in G_1 if and only if $f(v)$ is incident with $g(e)$ in G_2.

The graphs G_1 and G_2 are isomorphic if there exists an isomorphism from G_1 to G_2. We denote this $G_1 \cong G_2$.

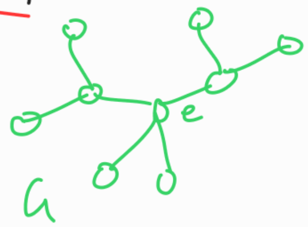
e.g. $C_4 \cong K_{2,2}$

Trees

A tree is a connected graph that has no cycles.

A forest is a graph that has no cycles.

e.g.



G is tree H is a forest

Let G be a graph with $e \in E(G)$.
We let $G \setminus e$ ("G delete e") denote the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$

e.g. $H = G \setminus e$ above.

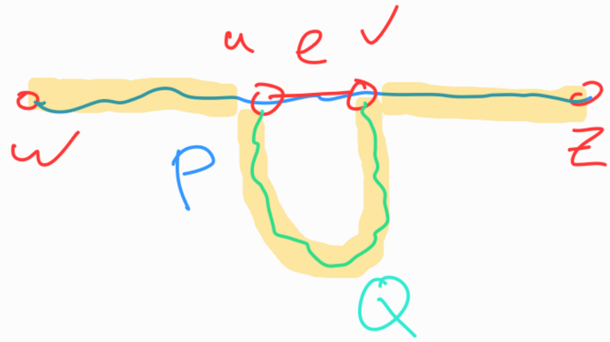
set difference:
for sets X and Y
$X \setminus Y = \{x \in X : x \notin Y\}$

The edge e is a <u>bridge</u> if $G \setminus e$ has more components than G. In particular, when G is connected, e is a bridge if $G \setminus e$ is disconnected.

<u>Thm</u> 2.1: Let G be a connected graph, with $e \in E(G)$. The edge e is a bridge if and only if e is not in any cycle of G.

<u>Proof</u>: ($\Rightarrow$) Suppose e is a bridge. Then $G \setminus e$ is disconnected, so has at least 2 components. So there exist vertices w and z in different components of $G \setminus e$, and there is no walk from w to z in $G \setminus e$. Thus every path from w

to z in G passes through e. Since G is connected, there is at least one such path P.



Let u and v be the ends of e. Suppose e is in a cycle. Then there is a path Q from u to v that avoids e. By replacing the u, e, v subpath of P with Q, we obtain a walk from w to z that avoids e. But no such walk exists. We deduce that e is not in a cycle.

($\Rightarrow$) For the other direction, see the course notes. $\square$