MATH361 | Lecture 6

Tutorial Friday 5:10pm
Assignment 1 due Monday 3pm

**Recap:** Let $G$ be a graph

For an edge $e \in E(G)$

$G \backslash e$ is the deletion of $e$ from $G$

$G / e$ is the contraction of $e$ from $G$

$e$ is a bridge if $G \backslash e$ has more components than $G$.

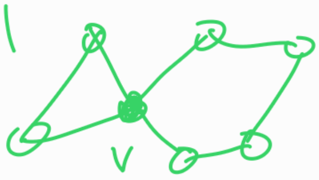
For a vertex $v \in V(G)$

$G - v$ is the deletion of $v$ from $G$

(this operation removes both $v$ and any edges incident with $v$).

**Defn:** $v$ is a cut vertex if $G - v$ has more components than $G$.

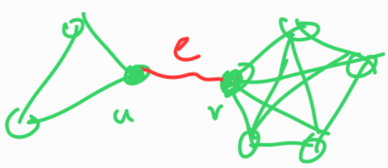
e.g. 1




$v$ is a cut vertex

(and this graph has no other cut vertices)

e.g. 2



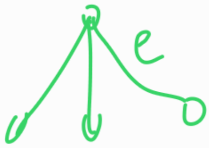

$u$ and $v$ are cut vertices

$e$ is a bridge.

**Question:** is it always the case that the ends of a bridge that is not a pendant edge are cut vertices?

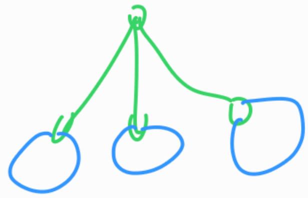
No...

for example




$K_{1,3}$

Still no...

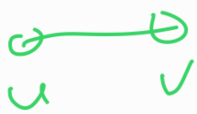


A vertex $v \in V(G)$ is a leaf if $v$ has degree 1.
An edge $e \in E(G)$ is a pendant edge if $e$ is incident with a leaf.

If $u$ is a vertex that is incident with a bridge, but $u$ is not a cut vertex, then either $u$ is a leaf, or $u$ becomes a leaf after deleting all incident loops.

Exercise 2.15: Prove this

Lemma 2.16: Let $G$ be a graph with at least 3 vertices and a bridge $e$ with ends $u$ and $v$. Then at least one of $u$ and $v$ is a cut vertex.

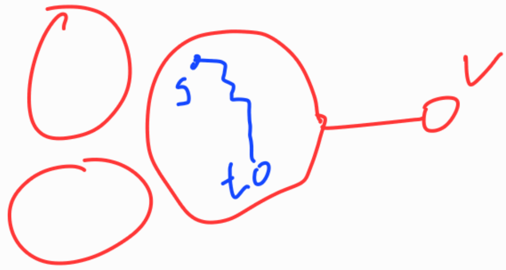



Lemma 2.14: Let $G$ be a graph and let $v \in V(G)$ be a leaf of $G$.

i) $v$ is not a cut vertex

ii) the pendant edge incident with $v$ is a bridge.

Proof of (i):

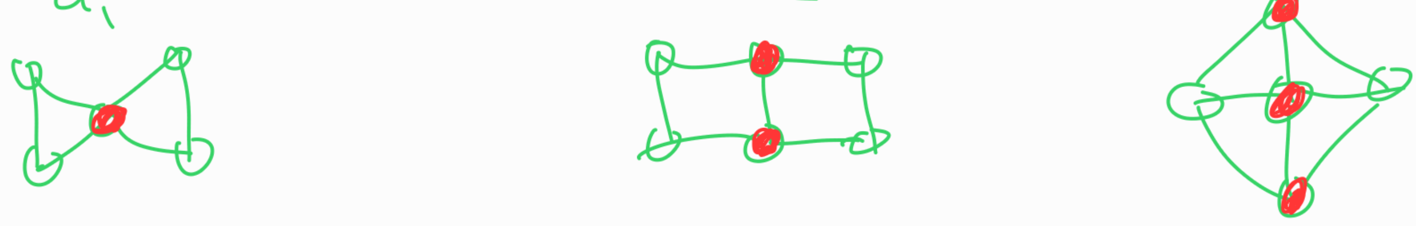



Since $v$ is a leaf, any path containing $v$ begins or ends at $v$. Let $s, t \in V(G) \setminus \{v\}$ such that $s$ and $t$ are in the same component of $G$. Then there is a path $P$ from $s$ to $t$. This path does not contain $v$ (by the above, and since $v \notin \{s,t\}$). Thus $P$ is also a path in $G-v$. Thus, there is a path between every pair of vertices in $G-v$ that are in the same component of $G$. Thus, $G-v$ has the same number of components as $G$, so $v$ is not a cut vertex. $\square$

## Connectivity

The following graphs are connected.

$G_1$ $\quad$ $G_2$ $\quad$ $G_3$



However, intuitively, they are "more well connected" as we move to the right.

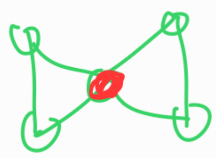
Let $G$ be a graph, and $X \subseteq V(G)$. We say $X$ is a vertex cut if $G-X$ has more components than $G$.

Note: cut vertex (single vertex) : $v \in V(G)$ such that $G-v$ has more components than $G$

vertex cut (set of vertices) : $X \subseteq V(G)$ such that $G-X$ has more components than $G$.

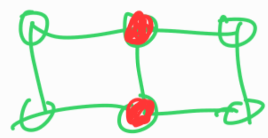
e.g.

$G_1$



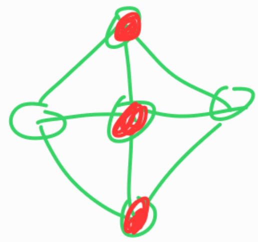
vertex cut ... of size 1

$G_2$



... of size 2

$G_3$



... of size 3.

No smaller vertex cut exists in each case.

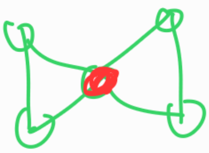
A k-vertex cut is a vertex cut of size k.

A vertex $v \in V(G)$ is a cut vertex iff $\{v\}$ is a 1-vertex cut

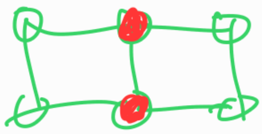
For $k \geq 2$, a graph is k-connected if it is connected, has at least $k+1$ vertices, and it has no vertex cuts of size less than k.

e.g.

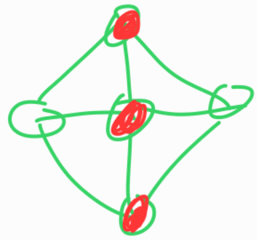
$G_1$



$G_2$



$G_3$



$G_1$ is connected

$G_1$ is not 2-connected

$G_2$ is 2-connected

$G_2$ is not 3-connected

$G_3$ is 3-connected

but not 4-connected

A graph is connected if it is 1-connected.

Note: this notion is sometimes called "vertex connectivity".

The connectivity of a graph $G$ is the largest integer $k$ such that $G$ is $k$-connected

e.g. $G_i$ has connectivity $i$.