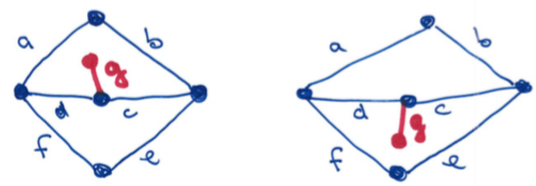
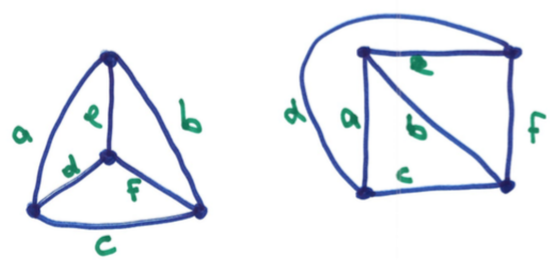
5. Planar Graphs II: Euler's Formula and Kuratowski's Theorem

Equivalent Planar Embeddings. Two planar embeddings of a graph G are *equivalent* if they have the same set of face boundaries; otherwise they are *inequivalent*.

Here is an example of two inequivalent embeddings of a graph:



On the other hand, here is an example of two equivalent embeddings of K_4:



Note that in one embedding abc is the outer face, while in the other cfd is the outer face. For the equivalence of planar embeddings, we only care about the set of all face boundaries: there is nothing special about the outer face.

> You might prefer to think of these embeddings as being equivalent or inequivalent on the sphere, where there is no distinguished outer face.

Inequivalent planar embeddings can be a pain. When can we guarantee that a planar graph has no inequivalent planar embeddings? As per usual, our strategy is to raise connectivity, but there is a catch.

Exercise 5.1. *Give an example of a 2-connected planar graph with inequivalent planar embeddings.*

Even 3-connectivity does not resolve things for us.

Exercise 5.2. *Give an example of a* 3*-connected planar graph with inequivalent planar embeddings.*

However, it turns out that the only problems with 3-connected graphs are caused by loops and parallel edges. We say that a graph G has a *unique planar embedding* if any two planar embeddings of G are equivalent.

Theorem 5.3 (Whitney, 1933)**.** *Let G be a simple* 3*-connected planar graph. Then G has a unique planar embedding.*

Proof. The proof is by induction on the number of edges of G. Since G is 3-connected, it has at least four vertices. Each vertex has degree at least three, so, by the Handshaking Lemma, G has at least six edges. If G has precisely six edges, then G is K_4, and one can check that the theorem holds in this case. Now assume that G has more than six edges and that the theorem holds for graphs with fewer edges than G.

[redacted] □

Two inequivalent embeddings G_1 and G_2 of a planar graph G can have different duals, where the underlying graphs of G_1^* and G_2^* are not even isomorphic. However, it follows from Theorem 5.3 that a simple 3-connected planar graph has a unique dual, as illustrated by the next corollary:

Corollary 5.4. *Let G be a simple* 3*-connected planar graph, and let G_1 and G_2 be planar embeddings of G. Then $G_1^* = G_2^*$.*

→ Now, by Corollary 5.4, given a simple 3-connected planar graph G, we can unambiguously refer to the dual G^* (which is the dual of the plane graph obtained using any planar embedding of G). In general (for a planar graph that may not be 3-connected and simple), we can only refer to the dual when we have a *plane* graph, which also describes the particular planar embedding.

Euler's Formula. This formula relates the number of edges, vertices and faces of a planar graph.

Let G be a plane graph. We let $v(G)$, $e(G)$, and $f(G)$ denote the number of vertices, edges, and faces of G respectively.

You may recall the proof of Euler's Formula from MATH161. There, the treatment was very informal; now we have a more secure foundation.

Theorem 5.5 (Euler's Formula)**.** *Let G be a connected plane graph. Then*

$$v(G) - e(G) + f(G) = 2.$$

Proof. The proof is by induction on the number of edges of G. If G is a connected graph with no edges, then G consists of a single isolated vertex and the theorem certainly holds. Assume then that G has at least one edge, and that the result holds for graphs with fewer edges than G.

Let e be an edge of G. First suppose that e is not a loop. Observe, by Lemma 4.11, that the faces of G/e are either faces of G whose boundary does not contain e, or faces of G whose boundary does contain e but with e removed. This means that

$$f(G/e) = f(G).$$

Also, since e is not a loop,

$$v(G/e) = v(G) - 1$$

and

$$e(G/e) = e(G) - 1.$$

The graph G is connected, so by Lemma 3.10, G/e is connected. By the induction assumption,

$$v(G/e) - e(G/e) + f(G/e) = 2.$$

Hence we have

$$(v(G) - 1) - (e(G) - 1) + f(G) = 2,$$

that is,

$$v(G) - e(G) + f(G) = 2.$$

On the other hand, when e is a loop, then $v(G/e) = v(G)$, $f(G/e) = f(G) - 1$, and $e(G/e) = e(G) - 1$. By a similar argument,

$$v(G/e) - e(G/e) + f(G/e) = 2,$$

so

$$v(G) - (e(G) - 1) + (f(G) - 1) = 2,$$

implying

$$v(G) - e(G) + f(G) = 2.$$

The result now follows by induction. $\square$

Euler proved this theorem in 1752, but he didn't do it in the setting of graphs. Rather, he proved it for polyhedra. It's not difficult to see that given a polyhedron, we can find a corresponding plane graph that preserves its edges, faces and vertices, and the incidences between these. However, not all connected plane graphs arise from polyhedra, so Theorem 5.5 is a little more general than Euler's original result.

Euler's Formula has many useful corollaries.

Corollary 5.6. *Let G be a connected planar graph. Then every planar embedding of G has the same number of faces.*

Proof. [redacted] □

Corollary 5.7. *Let G be a simple planar graph with at least three vertices. Then $e(G) \leq 3v(G) - 6$.*

Proof. [redacted] □

Corollary 5.7 is extremely useful. For example, we know that K_n has $n(n-1)/2$ edges; that is it has a *quadratic* number of edges. If n is large, then

$$n(n-1)/2 \gg 3n - 6$$

(that is to say, $n(n-1)/2$ is much larger than $3n-6$). Hence, when n is large, the vast majority of graphs on n vertices are not planar.

We can also use Corollary 5.7 to prove that every simple planar graph has a vertex of degree 5 or less. You may have seen, in MATH161, that this is a key ingredient towards proving that planar graphs are 5-colourable.

Also, Corollary 5.7 gives a useful way to prove that a graph is not planar.

Corollary 5.8. *K_5 is not planar.*

Proof. [redacted] □

Note, however, that Corollary 5.7 is not always helpful. Consider $K_{3,3}$. This graph has 6 vertices and 9 edges, and $9 \leq 3 \cdot 6 - 6 = 12$. Corollary 5.7 is not a characterisation of planarity (it is not an "if and only if" theorem). It says that if $e > 3v - 6$, then the graph is not planar. If $e \leq 3v - 6$, then it tells us nothing.

Having said that, there is a cunning way to use Euler's formula to prove that $K_{3,3}$ is not planar.

Exercise 5.9. *Use Euler's formula to prove that $K_{3,3}$ is not planar.*

Wagner's Theorem. For graphs G and H, we say that G *has* H *as a minor* if G has a minor isomorphic to H.

We know that K_5 and $K_{3,3}$ are not planar. We also know that any minor of a planar graph is also planar. It follows that any graph that has K_5 or $K_{3,3}$ as a minor cannot be planar. Remarkably, the converse is also true: any non-planar graph must have either K_5 or $K_{3,3}$ as a minor.

Theorem 5.10 (Wagner 1937)**.** *A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.*

The goal of this section is to prove Wagner's Theorem. However, it is worth first reflecting on why this theorem is so fundamental.

We always had a clear way to prove that a graph was planar — simply find a planar embedding. But it's a more subtle thing to show that a graph is *not* planar. Euler's Formula helps at times, but there are plenty of non-planar graphs that satisfy Euler's Formula. With Wagner's Theorem, we have a clear strategy. For any graph G we can either find a planar embedding — proving that G is planar — or we can find that G has a K_5 or $K_{3,3}$ minor — proving that G is non-planar.

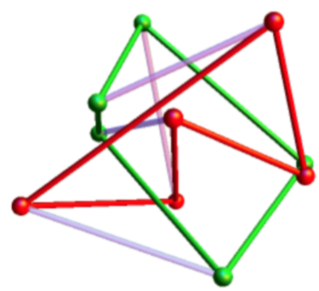
Of course, there still remains the algorithmic question. Can we design an algorithm to find a K_5- or $K_{3,3}$-minor efficiently, even when given a large graph? The short answer is that efficient algorithms are known, without being precise about what we mean by "efficient". The details, along with algorithmic issues related to finding embeddings on surfaces and finding minors, however fascinating, are (sadly) beyond the scope of this course.

There is another reason why Wagner's Theorem is amazing. It's truly a seminal theorem, in the sense that it is a theorem from which many other things have grown. To see why Wagner's Theorem is seminal we need to discuss the Graph Minors Project.

The Graph Minors Project. Let $\mathcal{G}$ be a class of graphs. Then $\mathcal{G}$ is *minor-closed* if whenever $G \in \mathcal{G}$ and H is a minor of G, then $H \in \mathcal{G}$.

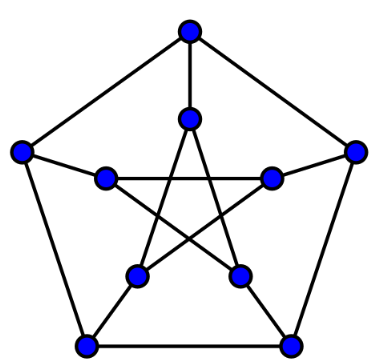
We have seen that planar graphs are a minor-closed class of graphs. There are many others. The graphs embeddable on any given surface — for example, the torus — form a minor-closed class. Forests form a minor-closed class; although not a particularly interesting one.

Here is a particularly cool example. A graph is *linklessly embeddable* if it can be embedded in $\mathbb{R}^3$ in such a way that no two cycles are *linked* (think of two rubber-bands that are entwined and cannot be separated without cutting one of them). Then the class of linklessly embeddable graphs is minor closed. In the graph below the red and green cycles are linked.



Exercise 5.11. *Prove that K_5 is linklessly embeddable.*

Exercise 5.12. *Prove that neither the Petersen Graph (see figure below) nor K_6 is linklessly embeddable.*



For a graph G, we say that H is a *proper* minor of G if H is a minor of G and $H \neq G$. Let $\mathcal{G}$ be a minor-closed class. Then an *excluded minor* for $\mathcal{G}$ is a graph

G that does not belong to $\mathcal{G}$ but has the property that all proper minors of G do belong to $\mathcal{G}$.

In this language, Wagner's Theorem says that K_5 and $K_{3,3}$ are the excluded minors for planar graphs, so that planar graphs are characterised by a list of two excluded minors. After a long series of difficult papers, over 20 years, the mathematicians Neil Robertson and Paul Seymour eventually proved:

Theorem 5.13. *Let $\mathcal{G}$ be a minor-closed class of graphs. Then $\mathcal{G}$ is characterised by a finite list of excluded minors.*

In other words there is a version of Wagner's Theorem for *every* minor-closed class of graphs. How long can the list of excluded minors be? Well that's another question. Consider graphs embeddable on the torus. It is known that there are at least 16,629 excluded minors for this class. No one knows how many more there are; but we do know the list is finite!

But we digress; let's return to Wagner's Theorem.

Proof of Wagner's Theorem. We will need a few lemmas before giving the proof of Wagner's Theorem.

Let G be a graph with a vertex v. Recall that the *neighbourhood* of v is the set of all vertices that are adjacent to v. We also say a vertex u in G is a *neighbour* of v if u and v are adjacent; that is, u is in the neighbourhood of v.

Lemma 5.14. *Let G be a loopless 3-connected planar graph. For any vertex v in G, the vertices in the neighbourhood of v are contained in a cycle.*

Proof. Since G is planar, it has a planar embedding. Let H be a plane graph obtained from G together with any planar embedding, and let v be a vertex of H. Then $H-v$ is loopless and 2-connected, by Lemma 3.18. By Theorem 4.13, the boundary of each face of $H-v$ is a cycle. Let f be the face of $H-v$ in which the vertex v was situated. Then the boundary of f in $H-v$ is a cycle, C say, where C is also a cycle of H, and each neighbour of v in H is in C. □

Let G be a graph with a separation $\{A,B\}$ of order 2 whose boundary is $\{u,v\}$. Recall that $G[A]$ is the induced subgraph of G on vertex set A. We define the graph G_A to be the graph obtained from $G[A]$ by adding an edge joining u and v. Similarly, we define G_B to be the graph obtained from $G[B]$ by adding an edge joining u and v.

Lemma 5.15. *Let G be a 2-connected graph with a proper separation $\{A,B\}$ of order 2. Then both G_A and G_B are isomorphic to proper minors of G.*

Proof. Say the boundary of $\{A, B\}$ is $\{u, v\}$. Let E_A be the edge set of $G[A]$ and let E_B be the edge set of $G[B]$. By symmetry, it suffices to show that G_A is isomorphic to a proper minor of G. Note that the difference between $G[A]$ and G_A is an extra edge joining u and v. Loosely speaking, we need to show that we can delete or contract each edge of $E(G) \setminus E_A$ in a way that we introduce this extra edge, thereby obtaining a minor isomorphic to G_A.

Say $w \in B \setminus A$. As G is 2-connected, $G - u$ is connected, by Lemma 3.18, so there is a path from w to v in $G - u$. All edges of this path are in $E(G) \setminus E_A$. Similarly, there is a path from w to u that uses only edges in $E(G) \setminus E_A$. Combining these paths we see that there is a walk — and hence a path P — from u to v that uses only edges in $E(G) \setminus E_A$. We now obtain the required edge by contracting all but one edge of P and deleting the other edges in $E(G) \setminus E_A$. □

Lemma 5.16. *Let G be a 2-connected graph with a separation $\{A, B\}$ of order 2. If both G_A and G_B are planar, then G is planar.*

A genuine rigorous proof of this lemma would require more topological techniques than we have available to us. For intuition, however, take planar embeddings of G_A and G_B where the edge uv is on the outer face. Glue these embeddings together along the edge $e = uv$. Voila! When we delete the edge e we obtain a planar embedding of G.

We are ready to prove Wagner's Theorem. We restate it here for convenience.

Theorem 5.10. A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a minor.

Proof. Assume that G is planar. By Corollary 4.12, every minor of G is planar. Neither K_5 nor $K_{3,3}$ is planar. Hence G has neither K_5 nor $K_{3,3}$ as a minor.

The real task is to prove the converse. Assume that G is not planar. We need to prove that G has either K_5 or $K_{3,3}$ as a minor. The proof is by induction on the number of vertices of G. It is easily checked that the only non-planar simple graph on five vertices (or fewer) is K_5. Thus, when G has (at most) five vertices, it has K_5 as a minor. Assume now that G has more than five vertices and, for induction, that any non-planar graph with fewer vertices than G has K_5 or $K_{3,3}$ as a minor. We first handle the case where G is not 3-connected.

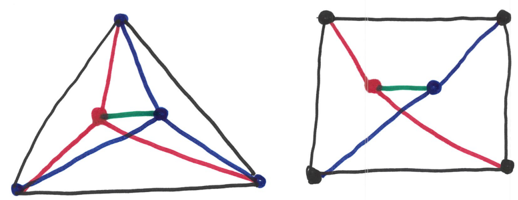
5.9.1. *If G is not 3-connected, then G has K_5 or $K_{3,3}$ as a minor.*

Subproof. Say that G is not 3-connected. Then it has a proper separation $\{A, B\}$ that is either of order 0, 1, or 2. Say that $\{A, B\}$ has order 0 or 1. Then $G[A]$ and $G[B]$ are minors of G with fewer vertices than G. If both $G[A]$ and $G[B]$

are planar, then it is easily seen that G is planar. Hence we may assume that $G[A]$ is non-planar, so has K_5 or $K_{3,3}$ as a minor by the induction assumption, so that G also has K_5 or $K_{3,3}$ as a minor.

If $\{A, B\}$ has order 2, then, by Lemma 5.15, both G_A and G_B are isomorphic to proper minors of G and have fewer vertices than G. If these graphs are planar, then, by Lemma 5.16, G is planar. As G is not planar one of them, say G_A, is not planar. Hence, by the induction assumption G_A, and hence also G, has K_5 or $K_{3,3}$ as a minor. $\lhd$

Now consider the case that G is 3-connected. Before we head into the details, it will be useful to keep in mind the following drawings of K_5 and $K_{3,3}$.



By Theorem 3.21, G has an edge $e = rb$ such that G/e is 3-connected. If G/e is not planar, then, by the induction assumption, it either has K_5 or $K_{3,3}$ as a minor, and so too does G. Hence we may assume that G/e is planar.

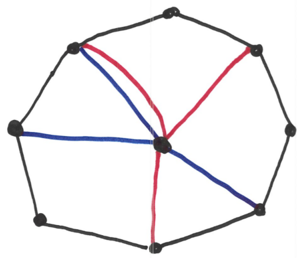
Let w denote the vertex that replaces $\{r, b\}$ in G/e. By Lemma 5.14, the neighbours of w are in a cycle C of G/e. Let R and B denote the set of edges of G incident with r and b, but excluding e, respectively. Thus $R \cup B$ is the set of edges incident with w in G/e. We will say that a vertex of C is *red* if it is incident with an edge in R and is *blue* if it is incident with an edge in B. Note that a vertex can be both red and blue. We say that a vertex of C is *coloured* if it is either red or blue.

The following diagram shows one possibility for the part of G/e near the vertex w. The outer cycle is C, the central vertex w, and we have coloured the edges incident with w using the above colour coding.

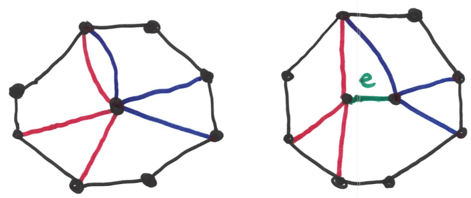
Both r and b have degree at least three in G, as otherwise G is not 3-connected. Hence $|R| \geq 2$ and $|B| \geq 2$.

5.9.2. *If C has at least four coloured vertices, then G has $K_{3,3}$ as a minor.*

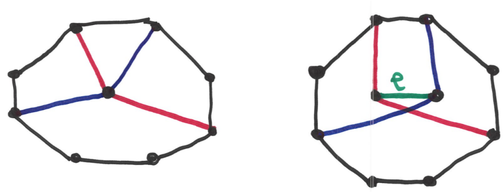
Subproof. There is a natural cyclic order on the vertices of C as given by the cycle. Consider the cyclic order induced by just the coloured vertices; i.e., the cyclic order where we traverse the cycle but skip any vertex that is not coloured.



(We still have the freedom to go in either direction, or start at any coloured vertex we choose.) Say that we have such a cyclic order $(v_1, v_2, \ldots, v_i, \ldots, v_t)$ where v_1 and v_i may be red, blue or both red and blue; $v_2, \ldots, v_{i-1}$ are red but not blue; and $v_{i+1}, \ldots, v_t$ are blue, but not red. Then it is readily checked that G is planar (for example, see the picture below).



Now assume C has at least four coloured vertices. As G is not planar, we do not have a cyclic order in the way described above. In this case, there exist indices j, k, l, m with $1 \le j < k < l < m \le t$ such that v_j and v_l are red, while v_k and v_m are blue.

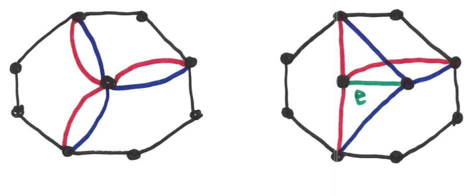


Delete all edges of G apart from the edges in C, the edges rv_j, bv_k, rv_l, bv_m and e, to obtain the graph illustrated on the right-hand side above. We may now contract surplus edges from C (incident with a vertex of degree 2) to obtain a $K_{3,3}$-minor. $\lhd$

5.9.3. *If C has exactly three coloured vertices, then G has K_5 as a minor.*

Subproof. Suppose the three coloured vertices v_1, v_2, v_3 are not all coloured both red and blue. Without loss of generality, the vertex v_2 is coloured red but not blue. Then (v_1, v_2, v_3) is a cyclic ordering as described in 5.9.2, that certifies that G is planar; a contradiction.

Hence all three are coloured both red and blue and we obtain a K_5-minor in a similar way to the earlier case for $K_{3,3}$. $\triangleleft$



To finish off, we note that if C has only one or two coloured vertices, then these vertices are a vertex cut in G of size at most two, contradicting the fact that G is 3-connected. Hence G either has $K_{3,3}$ or K_5 as a minor, as required. $\square$

Kuratowski's Theorem. Wagner's Theorem was proved in 1937. However, despite a somewhat different formulation, it can be seen to be equivalent to a related theorem proved by Kuratowski in 1930.

> Kuratowski actually wasn't the only one to prove this theorem: the same theorem was proved by Orrin Frink, Paul Smith and Lev Pontryagin at about the same time, although none of them published their proof.

Let G be a graph. Recall that a graph G' is a *subdivision* of G if G' can be obtained from G by subdividing edges. We say that a graph H is a *topological minor* of G if G contains a subgraph that is a subdivision of H. Usually we are just interested in topological minors up to isomorphism: we say that H *has* G *as a topological minor* if G contains a subgraph that is isomorphic to a subdivision of H.

Theorem 5.16 (Kuratowski's Theorem)**.** *A graph G is planar if and only if G does not have K_5 or $K_{3,3}$ as a topological minor.*

We will obtain Kuratowski's Theorem as a corollary of Wagner's Theorem. We need a few more bits of information. First, we observe that, loosely speaking, a topological minor is a specific sort of minor.

Lemma 5.17. *If H is a topological minor of G, then H is a minor of G.*

Proof. Say H is a topological minor of G. Then G contains a subgraph G' that is a subdivision of H. We now can contract edges from G' (corresponding to subdivided edges of H) to obtain the graph H. So H is indeed a minor of G. □

Observe that the converse of Lemma 5.17 does not hold in general.

Exercise 5.18. *Give an example of graphs G and H such that G has H as a minor, but G does not have H as a topological minor.*

But the converse does hold for certain special graphs. A graph is *cubic* if every vertex has degree 3.

Theorem 5.19. *Let H be a cubic graph and let G be a graph that has H as a minor. Then G has H as a topological minor.*

Before the proof, we introduce a notion that can be thought of as the opposite of contraction. Let w be a vertex of a graph G. Then a *vertex split* at w is obtained as follows:

(i) Partition the non-loop edges of G incident with w into two parts E_u and E_v.
(ii) For each loop e of G incident with w, add e to either E_u, to E_v, or to both E_u and E_v. Call the resulting sets E'_u and E'_v.
(iii) Replace the vertex w with two new vertices u and v such that u is incident with the edges in E'_u and v is incident with the edges in E'_v.
(iv) Add a new edge e joining u and v.

There is not a unique way to do a vertex split as we can have many choices for the partition $\{E_u, E_v\}$ (as well as which sets any loops are added to). Nonetheless, it follows from the definition that $G = G'/e$ if and only if G' is obtained from G by a vertex split where e is the new edge e.

→ If we perform a vertex split on a loopless graph, we don't need to worry about (ii). For loops, they can be split in a way that they remain a loop (incident to either u, or to v) or that they are no longer a loop (they have an end in u and an end if v).

Proof of Theorem 5.19. The graph G has a subgraph G' such that $H = G'/X$ for some set of edges X. We need to show that G' is a subdivision of H.

From the above discussion we know that we can obtain G' by a sequence of vertex splits from H. Let w be a vertex of H. Since w has degree 3, any partition $\{E_u, E_v\}$ of the edges incident with w will be such that one of the sets, has only one element, say f. In this case the vertex split simply subdivides the edge f. In other words a vertex split at a degree-3 vertex is a subdivision.

It is also easily seen that a vertex split at a degree-2 vertex is a subdivision. It follows that the sequence of vertex splits to obtain the graph G' from H is nothing more than a sequence of subdivisions. □

Note that $K_{3,3}$ is cubic. Hence we have:

Corollary 5.20. *If the graph G has $K_{3,3}$ as a minor, then G has $K_{3,3}$ as a topological minor.*

Of course, K_5 is not cubic. Nonetheless we still have:

Corollary 5.21. *If the graph G has K_5 as a minor, then G has K_5 or $K_{3,3}$ as a topological minor.*

Proof. Consider K_5. We first claim that any graph obtained from K_5 by splitting vertices either has $K_{3,3}$ as a minor or is a subdivision of K_5.

First consider a graph obtained from K_5 by a single vertex split at a vertex u. As u has degree 4, we can either partition the edges into two 2-elements sets, or into a singleton and a 3-element set. In the former case, it is easily checked that the graph resulting from this vertex split has $K_{3,3}$ as a minor (and after any subsequent vertex splits we also retain the $K_{3,3}$-minor, since these can be "reversed" by contracting an edge). In the latter case, the resulting vertex split corresponds to a subdivision. Note that this argument also applies to any subdivision of K_5. Hence it is indeed the case that the claim holds.

Now say that G has K_5 as a minor. Then G has a subgraph H that can be obtained from K_5 by a sequence of vertex splits. By the above claim, H is either a subdivision of K_5 or it has $K_{3,3}$ as a minor. In the former case, G has K_5 as a topological minor. In the latter case, by Corollary 5.20, H, and hence G, has $K_{3,3}$ as a topological minor. □

Proof of Kuratowski's Theorem. Assume that G is not planar. By Wagner's Theorem, G contains either K_5 or $K_{3,3}$ as a minor. By Corollaries 5.20 and 5.21 G has either K_5 or $K_{3,3}$ as a topological minor.

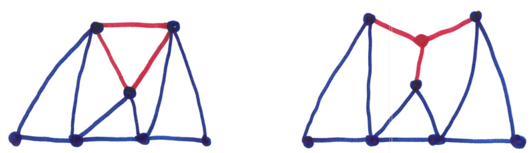
On the other hand, if G has either K_5 or $K_{3,3}$ as a topological minor, then, by Lemma 5.17, G has either K_5 or $K_{3,3}$ as a minor. Hence G is not planar. □

The ΔY exchange. This is an operation on graphs that arises in many surprising situations, for example in knot theory and theoretical physics. We first need some definitions. A *triangle* in a graph is a cycle with three edges. A *triad* is the set of edges incident with a vertex of degree 3, where none of them are parallel.

Let G be a graph with a triangle T on the vertices $\{u, v, w\}$. A ΔY exchange proceeds as follows:

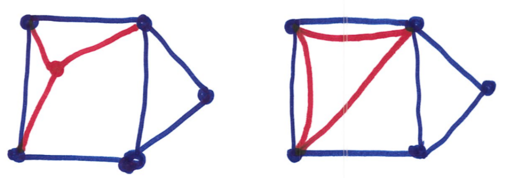
- Add a new vertex z, together with edges uz, vz, and wz.
- Delete the edges uv, vw and uw.

We say that the resulting graph is obtained from G by a ΔY *exchange on* T.



The $Y\Delta$ exchange is the reverse operation. Let u, v, w, z be vertices of G such that uz, vz, and wz form a triad. A $Y\Delta$ *exchange* proceeds as follows:

- Add edges uv, vw and uw.
- Delete the vertex z (along with its incident edges uz, vz, and wz).



Obviously a $Y\Delta$ exchange is just the opposite of a ΔY exchange. In essence we are switching a triad to a triangle and visa versa. Two graphs are said to be ΔY *equivalent* if one can be obtained from the other by a sequence of ΔY and $Y\Delta$ exchanges.

Exercise 5.22. *Prove that the Petersen Graph and K_6 are ΔY equivalent.*

How interesting was that! It turns out that there are, up to isomorphism, exactly seven graphs that are ΔY equivalent to K_6. This set of graphs is called the *Petersen Family*.

Why do we say "up to isomorphism"? If you start from K_6 and look at all the different graphs you can produce doing ΔY and $Y\Delta$ exchanges you will certainly get more than seven graphs; but quite a few are isomorphic to each other. If you forget about the labelling, then you get seven genuinely different graphs.

The next theorem was proved by Robertson, Seymour and Thomas in about 1992. The paper is available by **clicking here**.

Theorem 5.23. *A graph is linklessly embeddable if and only if it has no member of the Petersen family as a minor.*

→ In other words, the members of the Petersen family are precisely the excluded minors for the class of linklessly embeddable graphs!

Since we have introduced ΔY exchanges we have an opportunity to discuss another interesting class that arises in this context.

$Y\Delta Y$ reducible Graphs. A *series pair* in a graph is a pair of edges $e = uv$ and $f = vw$, where v has degree 2 and e and f are not parallel. Note that when an edge is subdivided, we get a series pair. A *series-parallel reduction* is any one of the following operations.

- Delete a loop.
- Contract a bridge.
- Delete a member of a parallel pair of edges.
- Contract a member of a series pair of edges.

Let G and H be graphs. Then G is $Y\Delta Y$ *reducible* to H if we can obtain H from G by a series of ΔY exchanges, $Y\Delta$ exchanges and series-parallel reductions.

We can now define an interesting class. A graph is $Y\Delta Y$ *reducible* if it is $Y\Delta Y$ reducible to a graph with no edges.

Exercise 5.24. *Prove that K_4 is $Y\Delta Y$ reducible.*

That's no coincidence. Indeed we have,

Theorem 5.25 (Epifanov 1966)**.** *Every planar graph is $Y\Delta Y$ reducible.*

Corollary 5.26. *Every simple planar graph with no bridges or series pairs either has a triad or triangle.*

It might be tempting to think that the $Y\Delta Y$-reducible graphs are just the class of planar graphs. But that's not true.

Exercise 5.27. *Prove that both* K_5 *and* $K_{3,3}$ *are* $Y\Delta Y$ *reducible.*

So much for that theory. But here is some interesting news.

Exercise 5.28. *Prove that no member of the Petersen family is* $Y\Delta Y$ *reducible.*

So $Y\Delta Y$-reducible graphs strictly contain planar graphs, but they do not contain everything. Certainly $Y\Delta Y$ reductions are not the same as taking minors, so the next theorem is also somewhat surprising.

Theorem 5.29 (Truemper 1989)**.** *The class of* $Y\Delta Y$*-reducible graphs is minor closed.*

Now that we know that the class is minor closed we can ask what the excluded minors for the class are. At one stage it was thought that maybe the excluded minors were just the members of the Petersen family, so that $Y\Delta Y$-reducible graphs and linklessly embeddable graphs were precisely the same class. But Neil Robertson came up with a counterexample to that conjecture. It turns out there are plenty of others.

Theorem 5.30 (Yaming Yi 2006)**.** *There are at least* $68,897,913,652$ *excluded minors for the class of* $Y\Delta Y$*-reducible graphs.*