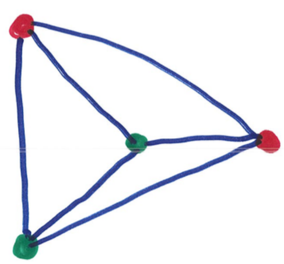
Let G be a graph. A *colouring* of G is a function $c : V(G) \to S$, where S is a set. Informally, we can think of the members of S as being colours, so that each vertex gets "coloured" by a member of S. If the set S has cardinality k, then we say that c is a k-colouring of G.

Although, for intuition, it is useful to think of the members of S as being colours, for mathematical arguments it is usually more convenient to take S to be the set $\{1, 2, \ldots, k\}$.

→ One could instead consider colouring the edges of a graph, which is known as an *edge colouring*; or the faces of a plane graph, which is known as a *face colouring*. For clarity, we can refer to a *vertex colouring*, rather than just a "colouring", to disambiguate between these notions.

Note that, so far, we haven't put any conditions on what colours we put where.

For example, is the following a 9-colouring of K_4?



Indeed it is. No one said that the function c described above had to be surjective/onto. In other words, in a k-colouring we do not have to use all the colours.

→ A 3-colouring is also a 4-colouring, and is also a k-colouring for all $k \geq 3$.

We now consider putting a restriction on our colourings.

A k-colouring is *proper* if whenever $e = uv$ is an edge then $c(u) \neq c(v)$. In other words, u and v must have different colours if they are connected by an edge. A graph G is *k-colourable* if it has a proper k-colouring.

It's easy to see that K_4, drawn above, is 4-colourable — just colour each vertex a different colour — so it is k-colourable for any integer $k \geq 4$. However, it is not 3-colourable. More generally, K_n is k-colourable if and only if $k \geq n$.

At times it is useful to visualise colouring in a different way. Let V be a set. We say that $(V_1, V_2, \ldots, V_k)$ is a *k-partition* of V if

(i) $V_1 \cup V_2 \cup \cdots \cup V_k = V$, and
(ii) For all $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$, we have $V_i \cap V_j = \emptyset$.

Note that we allow empty parts in a k-partition.

> What is the difference between a "k-partition" and a "partition" as seen previously? The main difference is, as just mentioned, we have no requirement that each V_i is non-empty. But we also have an order associated with the sets $V_1, \ldots, V_k$ in a k-partition (whereas a partition $\{V_1, V_2, \ldots, V_k\}$ with k parts is usually unordered).

A graph $G = (V, E)$ is *k-partite* if there is a k-partition $(V_1, V_2, \ldots, V_k)$ of V such that, for all $i \in \{1, 2, \ldots, k\}$, the graph $G[V_i]$ has no edges.

In other words, all the edges of the graph join different parts of the k-partition: if $e = uv$ is an edge, then there exists $i \neq j$ such that $u \in V_i$ and $v \in V_j$.

→ A bipartite graph is just another name for a 2-partite graph.

Lemma 6.1. *A graph is k-colourable if and only if it is k-partite.*

Proof. [redacted] □

The *chromatic number* of a loopless graph G is the minimum non-negative integer k for which G is k-colourable. Note that if G has a loop, then it is not k-colourable for any k, in which case we say it has chromatic number ∞.

Graph colouring is uninteresting for a graph with loops. For a loopless graph G with parallel edges, a proper k-colouring of G is also a proper k-colouring of the *simplification* of G (that is, the graph obtained from G by removing all but one edge in each parallel class). For this reason, we usually focus on simple graphs, when considering graph colouring.

It's easy to find a proper colouring of a (loopless) graph using lots of colours — just give each vertex a different colour. The goal is to do it with as few colours as possible. In other words the goal is to solve the problem of finding the chromatic number of a graph.

The mathematical formulation of graph colouring can hide the importance of potential applications.

Exercise 6.2. *A university wishes to timetable its examinations. To prevent the examination period from being too long it attempts to do this with as few time slots as possible. The exams for two courses cannot be scheduled at the same time if there is a student enrolled in both courses. Explain why this problem is equivalent to the problem of finding the chromatic number of a certain graph.*

Exercise 6.3. *A company manufactures a number of chemicals; say* $C_1, C_2, \ldots, C_n$*. Some chemicals cause explosions if they come into contact with each other. The company has a warehouse to store chemicals which it is planning to divide into rooms. Explosions, while indeed scientifically interesting, are perhaps best avoided. Hence we want to avoid storing incompatible chemicals in the same room. Explain why the problem of finding the minimum number of rooms is equivalent to the problem of finding the chromatic number of a graph.*

Due to the potential applications, the algorithmic question of finding the chromatic number of a graph is very interesting. A graph is 1-colourable if and only if it has no edges, so that's pretty easy. And 2-colourability is not hard either.

Exercise 6.4. *Find an efficient algorithm to decide if a graph is* 2*-colourable.*

We know that a graph is bipartite and hence 2-colourable if and only if it does not have any odd cycles. But a graph can have a lot of cycles. Checking every cycle seems like too much work. A good strategy for Exercise 6.4 must work differently.

Exercise 6.5. *How many cycles does* K_n *have?*

What about 3-colouring? It turns out that deciding whether a graph is 3-colourable is NP-complete. From an algorithmic point of view, colouring problems tend to be hard.

The chromatic number of a graph can be very high. To formalise an earlier observation:

Lemma 6.6. K_n *has chromatic number* n*.*

Proof. We cannot colour two distinct vertices the same colour as there is always an edge between them. So all vertices need different colours. Altogether we need n colours. $\square$

At times we can bound the chromatic number of a graph by using a structural feature, for example, the maximum vertex degree in the graph, typically denoted Δ.

Lemma 6.7. *Let G be a non-empty simple graph such that all vertices of G have degree at most Δ. Then G is $(\Delta + 1)$-colourable.*

Proof. We prove that G is $(\Delta + 1)$-colourable by induction on the number of vertices. If G has only one vertex, then this vertex has degree zero, and G is certainly 1-colourable, so the lemma holds in this case. Assume that G has at least two vertices and that the lemma holds for all simple graphs with fewer vertices than G.

Let v be a vertex of G. Consider $G - v$. As all vertices of $G - v$ have degree at most Δ, by the induction assumption $G - v$ is $(\Delta + 1)$-colourable. Consider a proper $(\Delta+1)$-colouring of $G-v$. This gives us a proper colouring of G except that v does not have a colour. The vertex v has degree at most Δ, so there are at most Δ colours used by the vertices adjacent to v. Hence there is always a colour that we can choose for v to obtain a proper Δ-colouring of G. $\square$

$\rightarrow$ This tells us, for example, that all simple cubic graphs are 4-colourable.

But, in fact, Brooks proved that we can usually do better than this.

Theorem 6.8 (Brooks' Theorem 1941)**.** *Let G be a simple connected graph whose vertices have degree at most Δ. If G is not an odd cycle or a complete graph, then G is Δ-colourable.*

Proof. We say that two vertices u, u' in G have distance two, and write $d(u, u') = 2$, if the shortest path between u and u' has length two. First we will show the following:

6.8.1. *If G has vertices x and z such that $d(x, z) = 2$ and $G - \{x, z\}$ is connected, then G is Δ-colourable.*

Subproof. Suppose that G has such vertices x and z, and let v_1 be a common neighbour of x and z. Consider an ordering $(v_1, v_2, \ldots, v_n)$ of the vertices of $G - \{x, z\}$ such that for each $i \in \{2, 3, \ldots, n\}$, the vertex v_i has a neighbour in $\{v_1, \ldots, v_{i-1}\}$. (Such an ordering can be obtained by a breadth-first search or a depth-first search, starting from v_1, for example.) We obtain a proper Δ-colouring for G as follows. We first assign both x and z colour 1. We then colour v_i, for $i = n, n-1, \ldots, 2$, in this order, as follows. The vertex v_i has degree at most Δ, and at least one neighbour in the set $\{v_1, \ldots, v_{i-1}\}$. Therefore, when we assign v_i a colour, it has at most $\Delta - 1$ neighbours that have already been assigned a colour. There are Δ colours available, so we can arbitrarily choose one of the unassigned colours. Finally, only v_1 has no colour assigned. But v_1 has at most Δ neighbours, and two of these neighbours, x and z, have been assigned the same colour, colour 1. So we can also colour v_1 an arbitrary colour that has not been assigned to its neighbours. $\triangleleft$

Suppose that G is 3-connected. If G is not a complete graph, then it has a pair of non-adjacent vertices, so it has a pair of vertices x and z at distance two (to see this, take a shortest path between any non-adjacent vertices, and pick the first and third vertices in this path). Since G is 3-connected, $G - \{x, z\}$ is connected, so G is Δ-colourable, by 6.8.1.

Next, suppose that G is 2-connected but not 3-connected. Assume that G is not an odd cycle. If G is an even cycle, then all vertices have degree at most 2, and G is 2-colourable. So we may assume that G is not a cycle. Then G has a vertex x of degree at least three. Suppose x is adjacent to every other vertex of G. Then, since G is not 3-connected, there are non-adjacent vertices $u, v \in V(G) \setminus \{x\}$. But then $d(u, v) = 2$ and $G - \{u, v\}$ is connected, so G is Δ-colourable by 6.8.1. So we may assume that x is not adjacent to every other vertex of G. Now, if $G - x$ is 2-connected, then there is a vertex z such that $d(x, z) = 2$ and $G - \{x, z\}$ is connected, hence G is Δ-colourable by 6.8.1. If $G - x$ is not 2-connected, then $G - x$ has distinct blocks B_1 and B_2 such that, for $i \in \{1, 2\}$, the block B_i contains precisely one cut vertex c_i. Since G is 2-connected, x has a neighbour $v_i \in V(B_i) \setminus \{c_i\}$, for $i \in \{1, 2\}$, where v_1 and v_2 are non-adjacent. By setting $\{x, z\} = \{v_1, v_2\}$, G is Δ-colourable by 6.8.1.

Finally, suppose that G is not 2-connected. Then it is easily seen that G is Δ-colourable if each block of G is Δ-colourable. (Why?) A block B of G is certainly Δ-colourable if it is not a complete graph or an odd cycle. Suppose G has a block B that is an odd cycle or complete graph. Then, as G is not 2-connected, B is not the only block of G, implying that B contains a cut vertex v of G. The vertex v has at least one neighbour outside of B, so Δ is greater than the maximum degree of B. It follows that B is Δ-colourable. Hence G is Δ-colourable. $\square$

$\rightarrow$ That means that every cubic graph except K_4 is 3-colourable.

While graphs with low vertex degrees are guaranteed to have low chromatic number, having high vertex degrees tells us nothing. For example, each vertex of $K_{50,50}$ has degree 50, but $K_{50,50}$ has chromatic number 2.

Having low vertex degrees is a structural feature that guarantees low chromatic number. Historically, the structural feature that has captured the most attention with respect to colouring is that of *planarity*.

The Map Colouring Problem. In a letter written to William Rowan Hamilton in 1852, Augustus De Morgan (of De Morgan's Laws fame) wrote:

> A student of mine asked me today to give him a reason for a fact which I did not know was a fact — and do not yet. He says that if a figure be anyhow divided and the compartments differently coloured so that figures with any portion of common boundary lines are differently coloured — four colours may be wanted, but not more.

In our language, we are given a plane graph and asked to colour the faces using at most 4 colours in such a way that every edge is incident to two faces that receive different colours.

However, we have a problem if the graph has bridges. In this case, the bridge is only incident with one face. So, more correctly, the map colouring problem can be stated as:

Problem 6.9. *Prove that every plane graph with no bridges has a proper* 4*-face colouring.*

It's not hard to convert this problem into a vertex-colouring problem.

Lemma 6.10. *There is a bijection between the k-face-colourings of a plane graph G and the k-vertex-colourings of G^*.*

The vertex-colouring version of Problem 6.9 then becomes:

Problem 6.11 (The 4-Colour Conjecture)**.** *Every loopless planar graph has a proper* 4*-colouring.*

The 4-Colour Conjecture is now a theorem, but it was hard won. It has a particularly interesting history. In about 1880 both Tait and Kempe published "proofs". However, in 1946 Tutte discovered a graph that demonstrated that Tait's proof was fatally flawed.

Kempe's proof did not survive as long as Tait's. Already in 1890 Heawood noticed a serious flaw. The good news was that Kempe's proof could be adapted to prove the 5-Colour Theorem, which we shall soon see.

Finally, in 1976, Appel and Haken published a proof of the Four Colour Conjecture. The proof had two parts. One is a long mathematical analysis that, in essence, reduced the problem to a massive case check. The other is a computer analysis that does the case check. This part is far too lengthy to ever do by hand.

Appel and Haken's proof was quite controversial. There were mathematical errors, but they all turned out to be inconsequential. Moreover, at the time, the idea of even using computers in proofs was contentious. (These days, so long as the computation is reproducible on an independent platform, most mathematicians see no problem with computer-aided proofs.)

In 1993, Robertson, Sanders, Seymour and Thomas set about finding a cleaner proof of the Four Colour Theorem. Their proof resolves any doubts about the mathematical correctness, but it still relies on a lengthy computer analysis.

The Five Colour Theorem. Given how hard the proof is for four colours, it is surprising that we can get very close to the exact answer quite easily.

Theorem 6.12 (Heawood, Kempe, about 1900)**.** *Every loopless planar graph is* 5*-colourable.*

The key idea in the proof is to use the technique of *Kempe Chains*. Let G be a graph with a proper vertex colouring. Assume that the vertex u is coloured **R**. Let **B** be another colour used in the colouring. Then an **R**–**B** Kempe Chain at u is obtained as follows. Recolour the vertex u to **B**. Recolour any adjacent vertices which are coloured **B** to **R**. Now recolour any adjacent vertices which are coloured **R** to **B**. Proceed in this way until no new vertices need be recoloured.

The next lemma essentially follows from the definition of Kempe Chain.

Lemma 6.13. *Let G be a graph with a proper colouring, and let $u \in V(G)$ such that u is coloured* **R**. *Say G is recoloured using an* **R**–**B** *Kempe Chain at u. Then,*

(i) *the resulting colouring is a proper colouring, and*
(ii) *the colour at a vertex $v \in V(G)$ is changed if and only if there is a path from u to v whose vertex colours alternate between* **R** *and* **B**.

The next lemma follows easily from Corollary 5.7.

Lemma 6.14. *Every simple non-empty planar graph has a vertex of degree at most five.*

Proof. [redacted] □

Proof of Theorem 6.12. Let G be a loopless planar graph. We proceed by induction on the number of vertices of the graph G. If G has zero or one vertices, then the result is clear. Assume that G has more than one vertex and that all loopless planar graphs with fewer vertices than G are 5-colourable.

By Lemma 6.14, G has a vertex v of degree at most five. Consider $G - v$. By the induction assumption, this graph is 5-colourable. Consider a 5-colouring of $G - v$ using the colours **R**, **O**, Y, **G** and **B**.

As G is planar we may take a planar embedding of G, and use the above colouring to colour all the vertices of G except v. If v has degree less than 5, then there are at most four neighbours of v, and we always have a colour available to colour v.

Hence we may assume that v has degree 5. Say the neighbours of v are in the cyclic order $(v_1, v_2, v_3, v_4, v_5)$. It may be that two of the neighbours of v have the same colour. Again, in this case, there are at most four colours at a vertex adjacent to v and we have a colour available for v.

The crunch comes when we see five different colours at the neighbours of v. We may assume that v_1 is coloured **R**, v_2 is coloured **O**, v_3 is coloured Y, v_4 is coloured **G** and v_5 is coloured **B**.

To get out of the fix, we use Kempe Chains to recolour the graph. Begin with a **R**-Y Kempe Chain starting at v_1 and use it to recolour $G - v$. If v_3 does not get recoloured then we no longer have any vertex in $\{v_1, v_2, v_3, v_4, v_5\}$ coloured **R**, and we have a colouring in which only four colours appear at a neighbour of v. In this case we may colour v by **R**.

Unfortunately it may be the case that we do recolour v_3, and all we have done is switch the colours **R** and Y on v_1 and v_3. But now we know that there is a (v_1, v_3)-path P, where all of the vertices of P are coloured **R** or Y. Moreover, by appending the path along v_1, v, v_3 to P, we obtain a cycle C. Consider the vertices v_2 and v_4. Note that, as the vertices v_1, v_2, v_3, v_4, v_5 appear in the cyclic order around v, it cannot be the case that v_2 and v_4 are both in the interior or both in the exterior of C.

It follows that any (v_2, v_4)-path in $G - v$ must use a vertex in the above path. That is, it must use a vertex coloured **R** or Y. From this we conclude that, if we recolour using an **O**-**G** Kempe Chain starting at v_2, the vertex v_4 does not get recoloured. After this recolouring we have no neighbour of v coloured **O** and we may therefore colour v to **O** to get a proper colouring of G. $\square$

So where did Kempe go wrong? We attempt to use the same strategy to prove the Four Colour Theorem. We find a vertex of G of degree at most 5. We have the same inductive hypothesis, except that this time we assume that $G - v$ is 4-colourable, and we consider a 4-colouring of $G - v$.

If v has degree less than 4, all is clear. If v has degree 4, then essentially the same Kempe Chain argument as in the proof of the Five Colour Theorem works. Thus we are left with the case that v has degree 5. The goal is to use Kempe Chains so that we only see three colours at the neighbours of v. You can find a presentation of Kempe's argument **here**.

Can you find the error in Kempe's argument?

Tait's Formulations of the 4-Colour Theorem. What about Tait? How did his proof fail?

What Tait did was pretty interesting. He reformulated the 4-Colour Conjecture as an edge-colouring problem.

A *k-edge colouring* of a graph G is an assignment of k colours to the edges of G. The colouring is *proper* if no two adjacent edges have the same colour. A graph is *k-edge-colourable* if it has a proper k-edge-colouring. Just as with vertex colourings, edge colourings arise in a variety of situations.

Tait noticed that the Four Colour Conjecture was equivalent to an edge-colouring problem. Recall that a *cubic* graph is one where every vertex has degree 3.

Theorem 6.15 (Tait 1880)**.** *Every loopless planar graph is 4-colourable if and only if every 3-connected cubic planar graph is 3-edge-colourable.*

Tait's theorem is an interesting theorem in its own right.

Before that, we take the opportunity to understand where Tait went wrong in his "proof" of the Four Colour Conjecture. First he — quite correctly — proved Theorem 6.15. His strategy then was to try to prove that all 3-connected cubic planar graphs are 3-edge-colourable.

He proceeded as follows. Recall that a *Hamiltonian cycle* in a graph is a cycle that uses all the vertices of the graph. He noticed that if we are looking for a proper 3-colouring of the edges of a graph, then having a Hamiltonian cycle is good.

Lemma 6.16. *If a cubic graph has a Hamiltonian cycle, then it is 3-edge-colourable.*

Proof. Let $G = (V, E)$ be a cubic graph with a Hamiltonian cycle. We first show that $|V|$ is even. Since every vertex has degree 3 we have

$$\sum_{v \in V} d(v) = 3|V|.$$

By the Handshaking Lemma $3|V| = 2|E|$. Hence $3|V|$ is even, so that $|V|$ must also be even.

Let C be a Hamiltonian cycle of G. Since $|V|$ is even, C has an even number of vertices and edges. Thus, there is a proper 2-edge-colouring of C, using the colours **R** and **B** say. As every vertex of G is a vertex of C, every vertex of G has a **R** and a **B** edge incident with it.

Consider the remaining edges. Colour these **G**. If the resulting colouring is not proper, then we must have two adjacent **G** edges. Say the vertex v is incident with both these edges. Then v has at least two **G** edges, a **R** edge, and a **B** edge incident with it, which means it has degree at least 4. This contradicts the assumption that G is cubic. $\square$

Tait's goal was then to prove that every 3-connected cubic planar graph had a Hamiltonian cycle, and he did indeed produce such a "proof". Tait was sunk when Tutte discovered a cubic 3-connected planar graph with no Hamiltonian cycle.

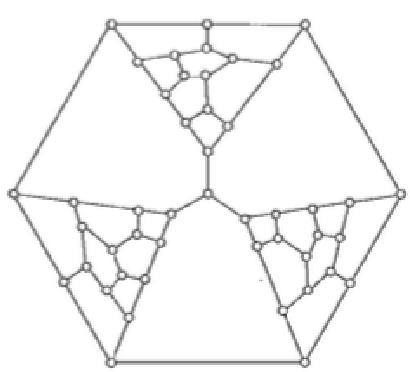


FIGURE 1. Tutte's Counterexample

But Tait did prove Theorem 6.15 which is pretty interesting. The next theorem is a key step.

Theorem 6.17. *A 3-connected cubic plane graph is 4-face-colourable if and only if it is 3-edge-colourable.*

Before proving the theorem, we recall that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is the group with elements $\{(0,0),(1,0),(0,1),(1,1)\}$ with addition defined coordinate-wise modulo 2. Hence $(1,0)+(1,1) = (0,1)$, $(1,1)+(1,1) = (0,0)$ etc. Note that, in $\mathbb{Z}_2 \times \mathbb{Z}_2$, we have $a+b=0$ if and only if $a=b$.

Proof. Let G be a 3-connected cubic plane graph. Note that, in particular, G has no bridges. Assume that G has a proper 4-face-colouring. For mathematical convenience our colours will be the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. We assign an edge colouring of G by assigning to each edge the sum of the colours of the faces incident with it. If a and b are the colours of the two faces incident with an edge e, then $a \neq b$, so that $a+b \neq 0$. Hence we have a 3-edge-colouring.

We need to show the 3-edge-colouring is proper. If a, b, and c are the values assigned to the three faces incident with a vertex v, then the edges incident with v have colours $a+b$, $b+c$, $c+a$. As $a \neq c$ we see that $a+b \neq b+c$. Similarly $b+c \neq c+a$ and $c+a \neq a+b$. Hence we have a proper 3-edge-colouring as required.

The proof of the converse is not too difficult, but it involves some additional technicalities and has been sacrificed. □

To derive Theorem 6.15 from Theorem 6.17 we will need a few lemmas. A plane graph is a *triangulation* if every face boundary (including the outer face) is a triangle (a cycle with three edges). It's not hard to see that we can add edges to a simple plane graph to turn it into a triangulation. From this we can prove:

Lemma 6.18. *Every loopless planar graph is 4-colourable if and only if every triangulation is 4-colourable.*

Proof. [redacted] □

We can easily observe:

Lemma 6.19. *If G is a triangulation, then G^* is a cubic graph.*

Proof. Since G is a triangulation, every face of G has degree 3. Hence every vertex of G^* has degree 3, so that G^* is cubic as required. □

Using Lemma 4.17(ii), we also have the following:

Corollary 6.20. *Let G be a connected plane graph. If G is cubic, then G^* is a triangulation.*

Yet again, connectivity plays a role. You might like to recall Lemma 5.15, which helps in the proof of the next lemma. The next lemma says that for 4-colouring triangulations, it suffices to focus on those that are 3-connected.

Lemma 6.21. *If every 3-connected triangulation is 4-colourable, then every triangulation is 4-colourable.*

Proof. [redacted] □

Recall that, by Corollary 5.4, for a simple 3-connected planar graph G, we can unambiguously refer to *the* dual G^*.

We omit the proof of the following lemma. Recall that Theorem 4.21 told us that the dual of a loopless 2-connected plane graph with at least three faces is also loopless and 2-connected. We also have:

Lemma 6.22. *If G is a simple 3-connected planar graph, then G^* is 3-connected and simple.*

We have all the ingredients ready for the proof of Theorem 6.15. We restate it below, before proving it.

Theorem 6.15. Every loopless planar graph is 4-colourable if and only if every 3-connected cubic planar graph is 3-edge-colourable.

Proof of Theorem 6.15. Assume that every loopless planar graph is 4-colourable. Then every triangulation is 4-colourable, by Lemma 6.18. By Corollary 6.20, the dual of a connected cubic planar graph is a triangulation. Hence, for every connected cubic planar graph, the dual is 4-colourable. So, by planar duality, every connected cubic planar graph is 4-face-colourable. In particular, this is true of 3-connected cubic planar graphs. So, by Theorem 6.17, every 3-connected cubic planar graph is 3-edge-colourable.

For the other direction, assume that every 3-connected cubic planar graph is 3-edge-colourable. By Theorem 6.17, every 3-connected cubic planar graph is 4-face-colourable. By Lemmas 6.19 and 6.22, the dual of a simple 3-connected triangulation is a simple 3-connected cubic graph. Hence, for every simple 3-connected triangulation, the dual is 4-face-colourable. This implies that every simple 3-connected triangulation is 4-colourable. Moreover, adding edges in parallel does not affect whether a graph is 4-colourable, so every 3-connected triangulation is 4-colourable. By Lemma 6.21, every triangulation is 4-colourable. Hence, by Lemma 6.18, every loopless planar graph is 4-colourable. □

Tait's Theorem is an example of an equivalent formulation of the 4-Colour Theorem. It turns out that there are many others. For example, Matyasdevich has shown that the 4-Colour Theorem is equivalent to the solvability of a diophantine equation involving thousands of variables! For a description of this, and a number of other formulations, see the 1998 paper of Robin Thomas **here**.

Hadwiger's Conjecture. One of the reasons that the 4-Colour Conjecture became so famous was that it is easily stated. A child can understand the problem. It's surprising that such a deceptively simple problem could turn out to be so difficult. In fact, graph theory is full of easily understood, but potentially very difficult, problems. Several of them concern colourings. Here is another well-known one. A loopless graph is *k-chromatic* if its chromatic number is at least k. (In other words, it cannot be coloured with fewer than k colours.)

Conjecture 6.23 (Hadwiger's Conjecture 1943)**.** *Let G be a loopless graph. If G is k-chromatic, then it has K_k as a minor.*

This conjecture, in fact, has an infinite number of cases. When $k = 1$ or $k = 2$, it is trivial. The case $k = 3$ is also not difficult.

Theorem 6.24. *Every loopless 3-chromatic graph has K_3 as a minor.*

Proof. Assume G is 3-chromatic. Then G is not bipartite, so G contains an odd cycle C. Since G has no loops, C has at least three edges. Deleting all edges not in C, and contracting all but three edges of C, gives the desired K_3-minor. □

The next case is also not too hard.

Theorem 6.25. *Every loopless 4-chromatic graph has K_4 as a minor.*

There are several ways to prove Theorem 6.25. One uses the following fact:

Lemma 6.26. *Every 3-connected graph has K_4 as a minor.*

Proof. [redacted] □

We sketch the proof of Theorem 6.25. Suppose G is a loopless 4-chromatic graph. It can be shown that if G is not 3-connected, it contains a (loopless)

3-connected minor H that is 4-chromatic. (To see this: if G is not 2-connected, and each block of G is k-colourable, then, by permuting the colours used in each block, we see that G is also k-colourable. Therefore, G has a 4-chromatic block. The argument is similar, but requires a little bit more care, in the case that this block is not 3-connected). As H is 3-connected, it has K_4 as a minor by Lemma 6.26, so G also has K_4 as a minor.

Life now starts to get interesting. In 1964 Wagner showed that the case $k = 5$ of Hadwiger's Conjecture was equivalent to the 4-Colour Conjecture. So that case is now a theorem.

The case $k = 6$ was proved by Robertson, Seymour and Thomas in 1993.

For $k \geq 7$, Hadwiger's Conjecture is entirely open. It's conceivable that someone comes up with a proof for the case $k = 7$ in the next 20 years, say, but it'd be a bit surprising.

The really interesting thing would be to find a way of solving all cases of Hadwiger's Conjecture in one fell swoop. But if anyone did this in the next 20 years, say, it'd be extremely surprising.

Arguably, what would not be as surprising is if someone answered Hadwiger's Conjecture in the negative. For higher values of k it might simply be false. All that has to be done to demonstrate this is to provide a counterexample. It's interesting to compare Hadwiger's Conjecture with another conjecture.

Conjecture 6.27 (Hajós's Conjecture 1950)**.** *Every k-chromatic graph contains a subdivision of K_k.*

In 1979, Caitlin found an 8-chromatic graph that contains no subdivision of K_8.

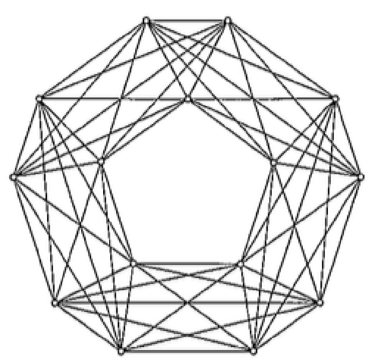


FIGURE 2. Caitlin's Graph

So much for Hajós's Conjecture. But the news is worse than that.

Typical and Rare Graph Properties. Consider a graph property P. The property may be “has an Eulerian walk”, “is planar”, or “is 4-colourable”. For each n, let P_n be the number of n-vertex graphs with property P, and let G_n be the number of simple graphs on the vertex set $\{1, 2, \ldots, n\}$. (Note that here we are counting the number of simple “vertex-labelled” graphs on n vertices, not the number of non-isomorphic graphs on n vertices.)

Exercise 6.28. *Calculate G_n explicitly.*

We will say that property P is *typical* if

$$\lim_{n\to\infty} \frac{P_n}{G_n} = 1.$$

And we will say that property P is *rare* if

$$\lim_{n\to\infty} \frac{P_n}{G_n} = 0.$$

Another way of saying this is that a property is typical if the probability that a randomly chosen graph has the property approaches 1 as n gets very large, and it is rare if the probability approaches 0.

Theorem 6.29 (Erdős and Fajtlowicz 1981)**.** *The property of being a counterexample to Hajós's Conjecture is a typical graph property.*

In other words, when n is very large, almost every graph on n vertices is a counterexample to Hajós's Conjecture.

But **György Hajós** was no fool. He was a member of an extremely strong group of Hungarian mathematicians that made major contributions to graph theory.

How could such a smart person be so wrong? The answer has to do with our *intuition* for what graphs are typically like. In the physical world we see things on the macroscopic scale. But the objects we see around us are, of course, made from much smaller particles. It's not obvious what that microscopic world is like. We have no intuition for that world, and our understanding of it has come about from hard-won scientific investigation.

On the other hand, we see graphs the other way around. Our intuition comes from looking at small graphs: ones we can draw. We naturally see graphs at the microscopic level. What goes on in the macroscopic scale — say for graphs with 10^{100} vertices — is not something that we have any natural intuition for.

Understanding what is typical in such large graphs is something we have to learn by hard-won mathematics.

Béla Bollobás, another Hungarian mathematician, proved that Hadwiger's Conjecture could not suffer the same fate as Hajós's Conjecture.

Theorem 6.30 (Bollobás 1980)**.** *Being a counterexample to Hadwiger's Conjecture is a rare property of graphs.*

In other words, if there are counterexamples to Hadwiger's Conjecture, they have to be few and far between.