MATH432 | Lecture 10

Assignment 2 due 15 April

Recap:

Corollary 3.20: Let G be a graph and $X \subseteq E(G)$

X is a cocircuit in $M(G) \iff X$ is a bond in G

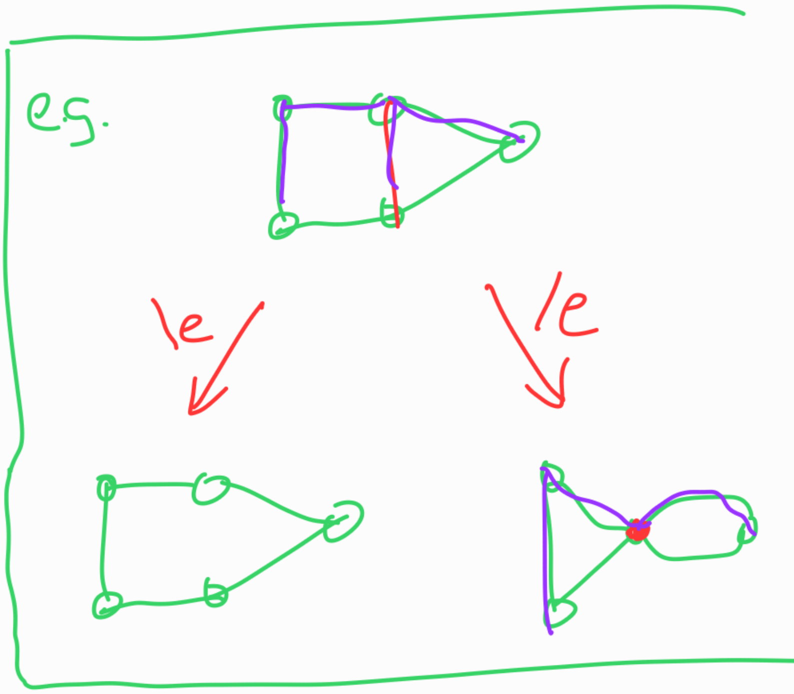
Theorem 3.26

If G is a plane graph, then $M^*(G) = M(G^*)$

Minors: For a graph G with an edge e, there are 2 natural ways to remove e:

* deletion, denoted $G \backslash e$

* contraction, denoted G/e



What happens to the forests of a graph when we delete or contract an edge?

F is a forest in $G \backslash e \iff F$ is a forest in G that doesn't contain e

F is a forest in $G/e \iff F \cup e$ is a forest in G

Definition Let M be a matroid and $e \in E(M)$.

$M \backslash e$ is the matroid on ground set $E(M) - e$ whose family of independent sets is

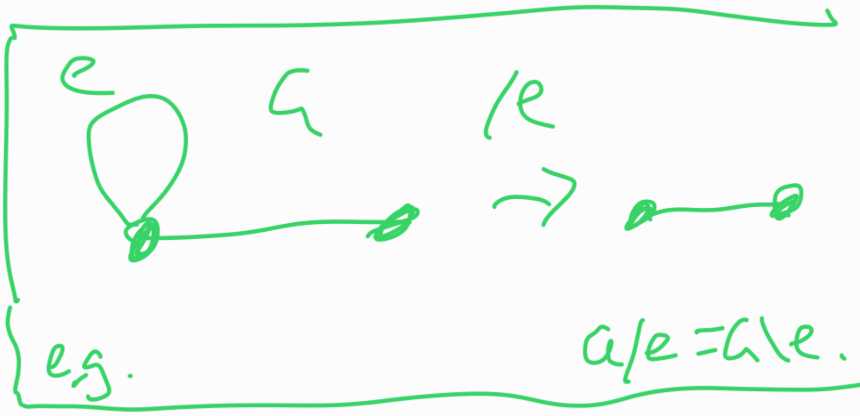
$$\{ I \subseteq E(M) - e : I \in \mathcal{I}(M) \}$$

M / e is the matroid on ground set $E(M) - e$ whose family of independent sets is

$$\{ I \subseteq E(M) - e : I \cup e \in \mathcal{I}(M) \}$$

when e is not a loop,

otherwise, when e is a loop, the independent sets are $\mathcal{I}(M \backslash e)$.



It is left as an exercise to check $M \backslash e$ and M/e are matroids. (check (I1) - (I3) hold).

Proposition 4.5: Let G be a graph and $e \in E(G)$.

Then $M(G)\backslash e = M(G\backslash e)$ and

$$M(G)/e = M(G/e)$$

Proof in online notes.

If a matroid N can be obtained from a matroid M by a (possibly empty) sequence of deletions and contractions, then N is a minor of M.

Def^n: For a class of matroids $\mathcal{M}$, we say $\mathcal{M}$ is minor-closed if, for any $M \in \mathcal{M}$, every minor of M is also in $\mathcal{M}$.

The last proposition implies that graphic matroids are a minor-closed class.

Example Let r and n are non-negative integers with $r \le n$ and $e \in E(U_{r,n})$.

$$U_{r,n}\backslash e = \begin{cases} U_{r,n-1} & \text{if } r < n \\ U_{n-1,n-1} & \text{if } r = n \leftarrow \text{every element is a coloop} \end{cases}$$

$$U_{r,n}/e = \begin{cases} U_{r-1,n-1} & \text{if } r \ge 1 \\ U_{0,n-1} & \text{if } r = 0 \leftarrow \text{every element is a loop} \end{cases}$$

In particular, uniform matroids are minor-closed.

Exercise: What is $r(M/e)$?

$U_{2,4}$

$$\mathcal{I}(U_{2,4}/e) = \{\{a\},\{b\},\{c\},\emptyset\}$$

$$r(M/e) = \begin{cases} r(M) - 1 & \text{if } e \text{ is not a loop} \\ r(M) & \text{if } e \text{ is a loop} \end{cases}$$

$$r(M\backslash e) = \begin{cases} r(M) & \text{if } e \text{ is not a coloop} \\ r(M) - 1 & \text{if } e \text{ is a coloop} \end{cases}$$

Proposition 4.4: Let e be a coloop in a matroid M. Then $M\backslash e = M/e$.

Minors and bases/circuits/rank

Proposition 4.13: Let M be a matroid and $e \in E(M)$

i) If e is not a coloop, then

$$\mathcal{B}(M\backslash e) = \{B \in \mathcal{B}(M) : e \notin B\}$$

ii) If e is not a loop, then

$$\mathcal{B}(M/e) = \{B \subseteq E(M) - e : B \cup e \in \mathcal{B}(M)\}$$

Proposition 4.14 Let M be a matroid and $e \in E(M)$

i) $\mathcal{C}(M \backslash e) = \{C \in \mathcal{C}(M) : e \notin C\}$

ii) The circuits of M/e are the minimal members of $\{C - e : C \in \mathcal{C}(M)\}$ when e is not a loop.

e.g.

Proposition 4.16 Let M be a matroid and $e \in E(M)$.

Then, for any $X \subseteq E(M) - e$,

i) $r_{M \backslash e}(X) = r_M(X)$

ii) $r_{M/e}(X) = r_M(X \cup e) - r_M(\{e\})$

Proof: (i) Let I be a maximal independent set of $M \backslash e$ contained in X. So $r_{M \backslash e}(X) = |I|$.

There is no independent set of M properly containing I and contained in X (otherwise it would be independent in $M \backslash e$, contradicting that I is maximal).

So $r_M(X) = |I| = r_{M \backslash e}(X)$ as required.

(ii) Suppose e is loop. Then $r_M(X \cup e) = r_M(X)$ and $r_M(\{e\}) = 0$, so $\underline{r_M(X \cup e) - r_M(\{e\}) = r_M(X)}$.

$*$

Also $M/e = M \backslash e$, so

$$\begin{aligned} r_{M/e}(X) &= r_{M \backslash e}(X) \\ &= r_M(X) \quad \text{by (i)} \\ &= r_M(X \cup e) - r_M(\{e\}) \quad (\text{by } *) \end{aligned}$$

as required

Suppose e is not a loop. Then $r_M(\{e\}) = 1$

Let I be a maximal independent set of M/e contained in X, so $r_{M/e}(X) = |I|$.

Then $I \cup e$ is independent in M.

Moreover $I \cup e$ is a maximal independent set contained in $X \cup e$ (for, if not, we can obtain a contradiction to the fact that I is maximal) so

$$\begin{aligned} r_M(X \cup e) &= |I \cup e| \\ &= |I| + 1 \\ &= r_{M/e}(X) + r_M(\{e\}) \end{aligned}$$

Rearranging shows (ii) holds. $\square$