MATH432 | Lecture 12

Recall: a class of matroids $\mathcal{M}$ is <u>minor-closed</u> if, for any $M \in \mathcal{M}$, and any minor N of M, we have $N \in \mathcal{M}$.

→ graphic matroids are minor-closed.

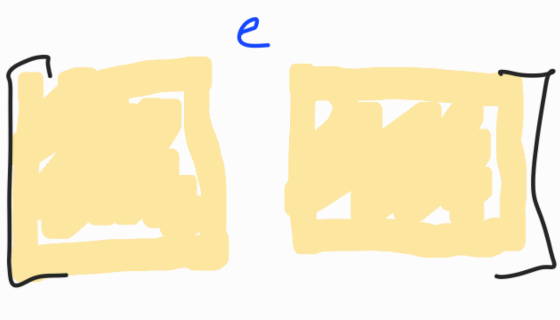
→ for a field $\mathbb{F}$, is the class of $\mathbb{F}$-representable matroids minor-closed?

<u>Lemma</u> 4.27: Let M be an $\mathbb{F}$-representable matroid, for some field $\mathbb{F}$, and let N be a minor of M. Then N is $\mathbb{F}$-representable.

<u>Proof:</u> It suffices to show that any single-element deletion or contraction of M is $\mathbb{F}$-representable.

Let $M = M[A]$ for some matrix A over $\mathbb{F}$, with $e \in E(M)$, so e labels a column of A.

Let A' be the matrix obtained from A by deleting the column labelled e. Then the linearly



independent sets of columns of A' are precisely the linearly independent sets of columns of A that don't contain the column labeled e. So $M\backslash e = M[A']$, demonstrating any single-element deletion is $\mathbb{F}$-representable.

Now $M/e = (M^*\backslash e)^*$ by Corollary 4.19.

M^* is $\mathbb{F}$-representable (since $\mathbb{F}$-rep'able matroids are closed under duality)

so $M^*\backslash e$ is $\mathbb{F}$-representable (by the foregoing)

so $(M^*\backslash e)^*$ is $\mathbb{F}$-representable.

This shows that any single-element contraction of M is $\mathbb{F}$-representable, and the result follows. □

Note: if a class is closed under duality and single-element deletions, then it is closed under minors.

Suppose that

$$\Big[\overbrace{\;I_r\;}^{X} \;\Big|\; \overbrace{\;D\;}^{Y}\Big] \quad \text{or} \quad X\left\{\Big[\overbrace{\;D\;}^{Y}\Big]\right.$$

represents M.

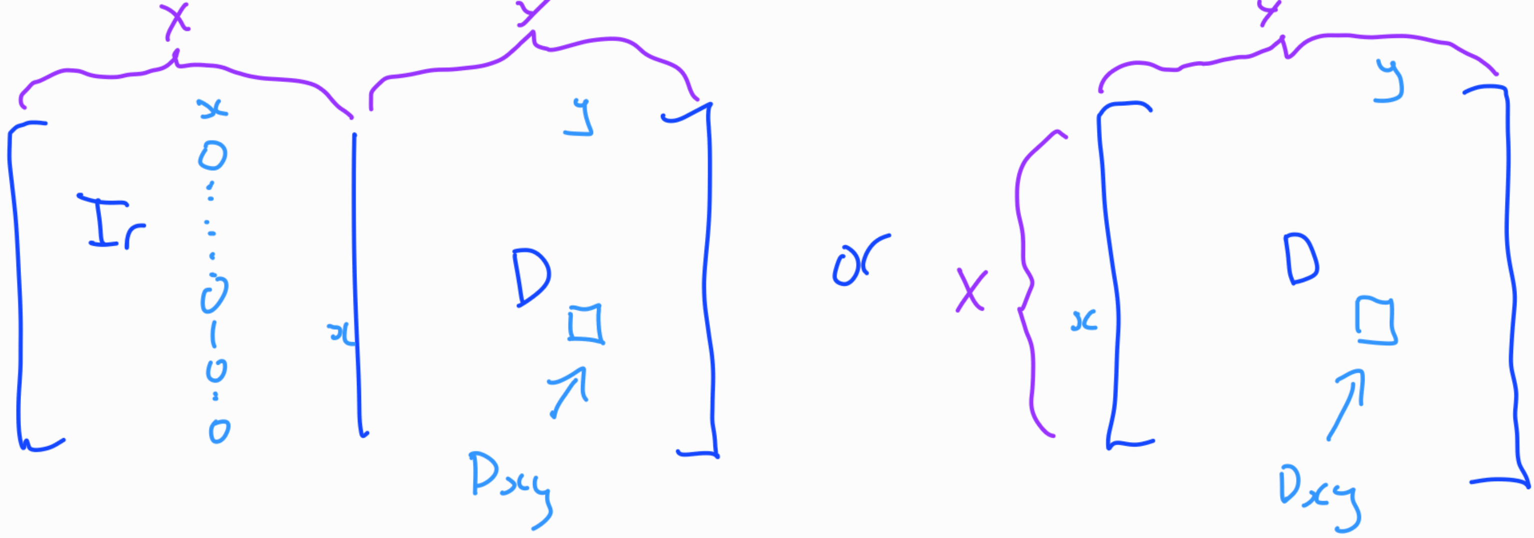
How can we find a reduced $\mathbb{F}$-representation of $M\backslash e$ or M/e (or one in standard form)?

To delete $e \in E(M)$ we can simply remove the corresponding column of $[I_r \,|\, D]$ for a representation of $M\backslash e$.

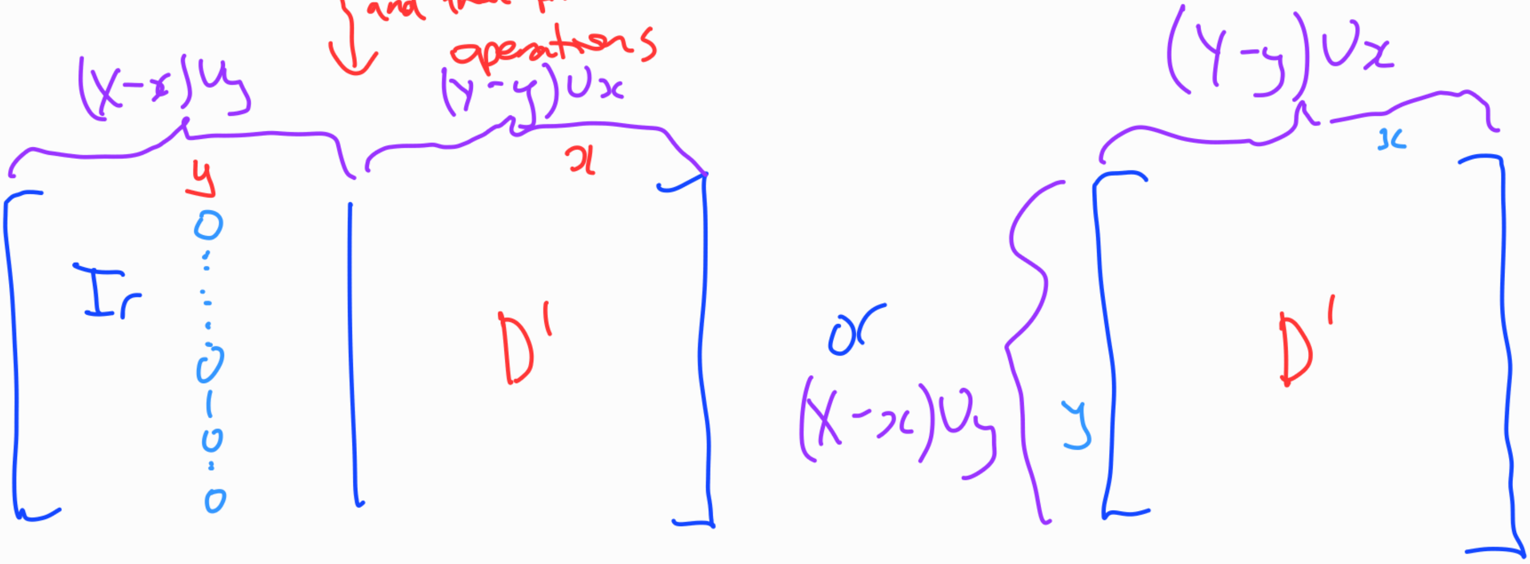
If $e \in Y$, we retain a representation in standard form, i.e we could just delete a column from the reduced representation.

If $e \in X$, to keep a representation in standard form, we first pivot to swap x out of the basis corresponding to I_r.

Pivoting: Let D_{xy} denote the entry of D in the x row and y column.



swap x and y columns and then perform row operations



We can pivot on xy when $D_{xy} \neq 0$.

Note: there is a non-zero entry in the y column whenever y is not a loop

$$\begin{array}{cc} x & y \\ \left[\; I_r \;\right] & \left[\begin{array}{c|c} D_{xy} & \underset{\sim}{c}^T \\ \hline \underset{\sim}{d} & D[X-x, Y-y] \end{array}\right] \end{array}$$

scale first row $\longrightarrow$

$$
\begin{array}{c}
\begin{array}{cc} x & \end{array} \\
\left[\begin{array}{c|c} D_{xy}^{-1} & \underset{\sim}{0}^T \\ \hline \underset{\sim}{0} & I_{r-1} \end{array}\right]
\end{array}
\begin{array}{c}
\begin{array}{cc} y & \end{array} \\
\left[\begin{array}{c|c} 1 & D_{xy}^{-1}\underset{\sim}{c}^T \\ \hline \underset{\sim}{d} & D[X-x, Y-y] \end{array}\right]
\end{array}
$$

subtract first row from other rows to get 0 in y column $\longrightarrow$

$$
\begin{array}{c}
\begin{array}{cc} x & \end{array} \\
\left[\begin{array}{c|c} D_{xy}^{-1} & \underset{\sim}{0}^T \\ \hline -D_{xy}^{-1}\underset{\sim}{d} & I_{r-1} \end{array}\right]
\end{array}
\begin{array}{c}
\begin{array}{cc} y & \end{array} \\
\left[\begin{array}{c|c} 1 & D_{xy}^{-1}\underset{\sim}{c}^T \\ \hline \underset{\sim}{0} & \begin{array}{c} D[X-x, Y-y] \\ -D_{xy}^{-1}\underset{\sim}{d}\,\underset{\sim}{c}^T \end{array} \end{array}\right]
\end{array}
$$

Now swap columns to obtain the pivot of D on xy denoted D^{xy}.

In summary:

$$
\begin{array}{cc}
 & \begin{array}{cc} y & \overbrace{\qquad\qquad}^{Y-y} \end{array} \\
\begin{array}{c} x \\ \scriptstyle X-x \left\{ \vphantom{\begin{array}{c}a\\a\\a\end{array}} \right. \end{array} &
\left[\begin{array}{c|c} D_{xy} & \underset{\sim}{c}^T \\ \hline \underset{\sim}{d} & D[X-x, Y-y] \end{array}\right]
\end{array}
\qquad\qquad
\begin{array}{cc}
 & \begin{array}{cc} x & \overbrace{\qquad\qquad}^{Y-y} \end{array} \\
\begin{array}{c} y \\ \scriptstyle X-x \left\{ \vphantom{\begin{array}{c}a\\a\\a\end{array}} \right. \end{array} &
\left[\begin{array}{c|c} D_{xy}^{-1} & D_{xy}^{-1}\underset{\sim}{c}^T \\ \hline -D_{xy}^{-1}\underset{\sim}{d} & \begin{array}{c} D[X-x, Y-y] \\ -D_{xy}^{-1}\underset{\sim}{d}\,\underset{\sim}{c}^T \end{array} \end{array}\right]
\end{array}
$$

$$D \qquad\qquad\qquad\qquad D^{xy}$$

Returning to deletion: to delete e when e

labels a row in a reduced representation, pivot on an entry in the e row, say ey, such that the ey entry is non-zero, then delete the e column after pivoting

To contract $e \in E(M)$, we dualise, delete, dualise. Recall that if

$$\begin{array}{cc} \overbrace{\quad x \quad}^{X} & \overbrace{\quad y \quad}^{Y} \end{array}$$
$$\left[\begin{array}{c|c} I_r & D \end{array}\right] \quad \text{or} \quad \begin{array}{cc} & y \\ x & \left[\begin{array}{c} D \end{array}\right] \end{array}$$

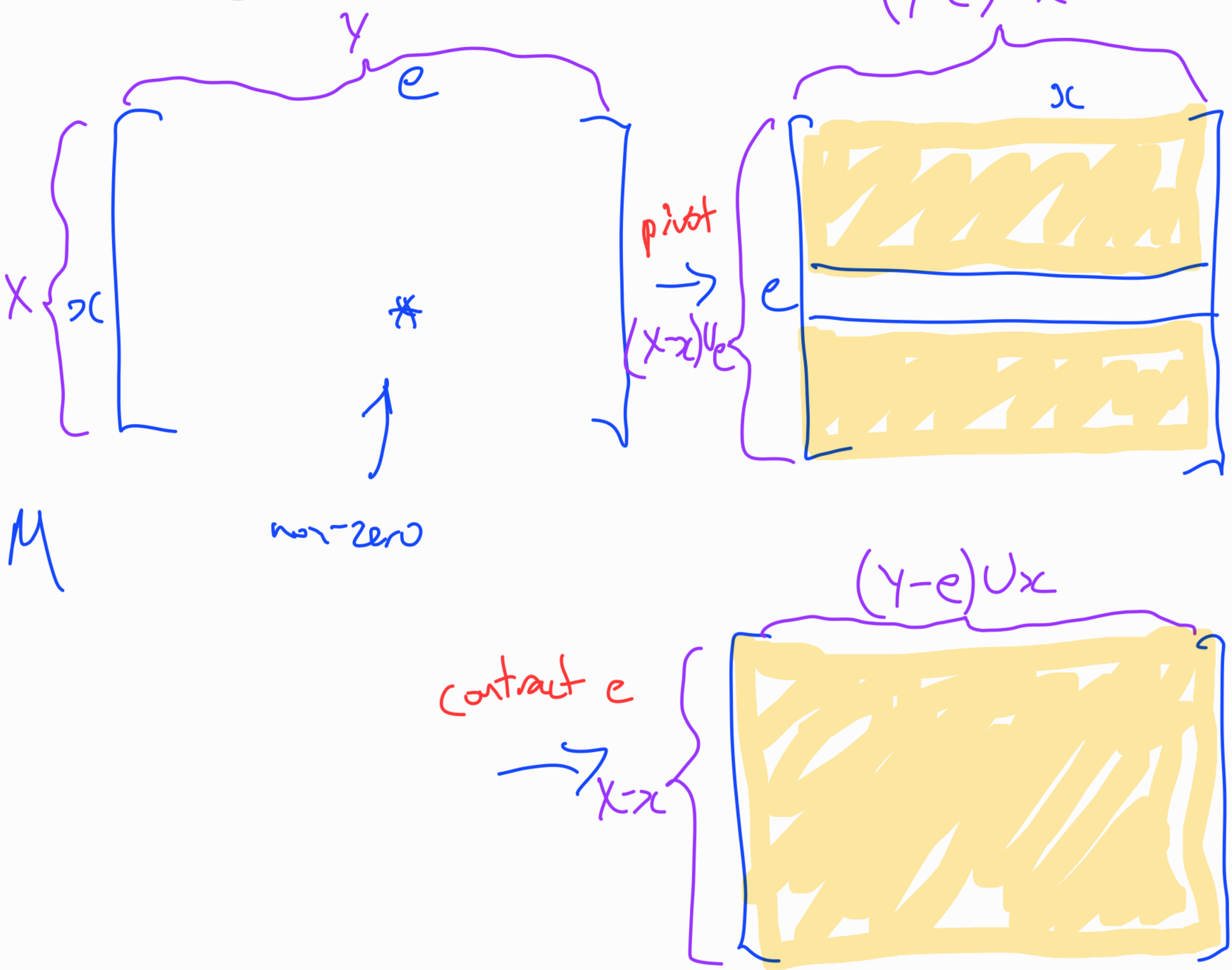
represents M, then

$$\begin{array}{cc} \overbrace{\quad\quad}^{Y} & \overbrace{\quad x \quad}^{X} \end{array}$$
$$\left[\begin{array}{c|c} I_{n-r} & D^T \end{array}\right] \quad \text{or} \quad \begin{array}{cc} & x \\ y & \left[\begin{array}{c} D^T \end{array}\right] \end{array}$$

represents M^*

Thus, if $e \in X$, then $M/e = (M^*\backslash e)^*$ has a reduced representation where we delete a row from D (since we delete a column from D^T).

To contract $e \in Y$, we first pivot so that e labels a row of the reduced representation of M, then we delete this row.



Closure and flats

Let M be a matroid on ground set E, and let $X \subseteq E$.

The closure of X is the set

$$\{e \in E : r(X \cup e) = r(X)\}.$$

We define the function (called the closure operator)

$$cl_M : 2^E \to 2^E$$

that, for any $X \subseteq E$, maps X to the closure of X.

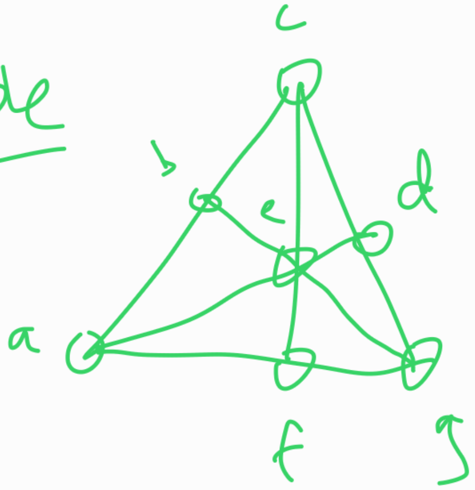
Note that for any $x \in X$, we have $x \in cl(X)$, so we can, equivalently, say

$$cl(X) = X \cup \{e \in E - X : r(X \cup e) = r(X)\}.$$

For an element $e \in cl(X)$, we say X spans e.

A set $X \subseteq E$ for which $cl(X) = X$ is called a flat of M

Example



This is called the non-Fano matroid denoted F_7^-

$$cl(\{a,b\}) = \{a,b,c\} = cl(\{a,b,c\})$$

$$cl(\{c\}) = \{c\}$$ so $\{c\}$ is a flat.

$$cl(\{a,b,f\}) = E(F_7^-)$$

Any singleton is a flat.

$E(F_7^-)$ is a flat

$\{a,b,c\}$ (and any of the 3-element rank-2 circuits) is a flat.