MATH4352 | Lecture 13

Assignment 3 due Thursday 2 May

Last time: closure and flats

For $X \subseteq E(M)$

$$cl_M(X) = \{e \in E(M) : r(X \cup e) = r(X)\}.$$

X is a flat if $cl(X) = X$.

Lemma 5.6(i) For a matroid M and $X \subseteq E(M)$,

$$r(cl(X)) = r(X).$$

Proof: Let B_X be a basis of X (i.e. a basis of $M|X$) and let $x \in cl(X) - X$.

$$r(B_X \cup x) \underset{(R2)}{\leq} r(X \cup x) = r(X) = |B_X| = r(B_X) \underset{(R2)}{\leq} r(B_X \cup x)$$

so equality holds throughout, in particular

$r(B_X \cup x) = |B_X| < |B_X \cup x|$. Thus $B_X \cup x$ is dependent, implying B_X is a basis of $cl(X)$.

So $r(cl(X)) = |B_X| = r(X)$ as required. $\square$

X is a spanning in M $\iff$ X contains a basis of M

$\iff r(X) = r(M)$

$\iff \mathrm{cl}(X) = E(M)$

X is a hyperplane in $M \iff X$ is a maximal non-spanning set in M

$\iff r(X) = r(M) - 1$ and

$r(X \cup e) = r(M) \quad \forall e \in E(M) - X$

$\iff X$ is a flat with $r(X) = r(M) - 1$

Properties of closure

(CL1) For $X \subseteq E$, $X \subseteq \mathrm{cl}(X)$

(CL2) For $X \subseteq Y \subseteq E$, $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$

(CL3) For $X \subseteq E$, $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$

(CL4) For $X \subseteq E$, $x \in E$,

if $y \in \mathrm{cl}(X \cup x) - \mathrm{cl}(X)$ then $x \in \mathrm{cl}(X \cup y)$.

(CL4) is also known as the Mac Lane-Steinitz exchange property.

Proposition 5.8 Let M be a matroid. Then the closure operator cl_M of M satisfies (CL1)-(CL4).

It turns out that, given a closure operator satisfying (CL1)-(CL4), there is a matroid having this closure operator (the independent sets are:

$$\mathcal{I} = \{X \subseteq E : x \notin cl(X - x) \text{ for all } x \in X\}$$)

<u>Lemma 5.7</u>: Let M be a matroid with $X \subseteq E(M)$ and $e \in E(M)$.

i) $r(X) \leq r(X \cup e) \leq r(X) + 1$.

ii) $r(X \cup e) = r(X) + 1 \iff e \notin cl(X)$.

<u>Proof</u>: Observe that

$$r(X) \underset{(R2)}{\leq} r(X \cup e) \underset{\substack{\text{by } (R3) \\ \text{when } e \notin X}}{\leq} r(X) + r(\{e\}) \underset{(R1)}{\leq} r(X) + 1.$$

which shows (i) holds.

Now $r(X \cup e) \in \{r(X), r(X)+1\}$

As $e \in cl(X) \iff r(X \cup e) = r(X)$, (ii) follows. $\square$

<u>Proof</u> of Prop 5.8: (CL1) is immediate from the defⁿ and (CL2) is left as an exercise.

We'll first prove (CL3).

First $cl(X) \subseteq cl(cl(X))$ by (CL1) and (CL2)

Now let $x \in cl(cl(X))$. Then

$$r(cl(X) \cup x) \underset{\uparrow \text{ Lemma 5.6i}}{=} r(cl(X)) = r(X) \underset{(R2)}{\leq} r(X \cup x) \underset{\substack{\uparrow \\ (CL1) \\ +(R2)}}{\leq} r(cl(X) \cup x)$$

So equality holds throughout, and $r(X \cup x) = r(X)$, implying $x \in cl(X)$.

This tells us $cl(cl(X)) \subseteq cl(X)$

Hence $cl(X) = cl(cl(X))$, i.e. (CL3) holds.

Now (CL4). Let $y \in cl(X \cup x) - cl(X)$.

Then $r(X \cup \{x, y\}) = r(X \cup x)$

and $r(X \cup y) = r(X) + 1$ by Lemma 5.7 as $y \notin cl(X)$.

$$\text{So } r(X)+1 = r(X \cup y) \underset{(R2)}{\leq} r(X \cup \{x,y\}) = r(X \cup x) \underset{\substack{\uparrow \\ \text{by Lemma 5.7i}}}{\leq} r(X) + 1$$

Equality holds throughout, so $r(X \cup y) = r(X \cup \{x, y\})$

implying $x \in cl(X \cup y)$ as req^d! $\square$

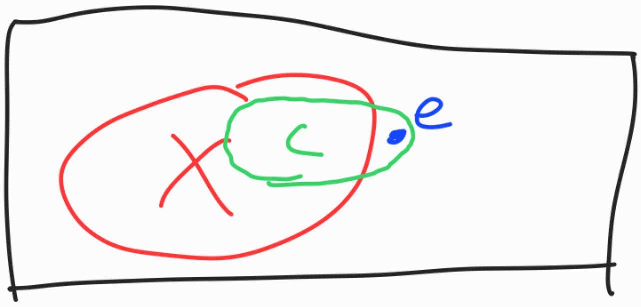
Note that, by (CL3), for any $X \subseteq E(M)$

$cl(X)$ is a flat.

<u>Proposition 5.9</u> Let M be a matroid, $X \subseteq E(M)$ and $e \in E(M) - X$. Then

$e \in cl(X)$ if and only if there is a circuit C of M that contains e and is contained in $X \cup e$.

Proof in online notes.



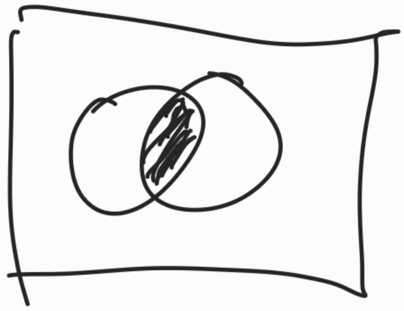
<u>Proposition 5.10</u> Let M be a matroid and let F_1 and F_2 be flats of M.

Then $F_1 \cap F_2$ is a flat of M.

<u>Proof</u>: Suppose $F_1 \cap F_2$ is not a flat.

Then $cl(F_1 \cap F_2) \neq F_1 \cap F_2$, so there exists an element $e \in cl(F_1 \cap F_2) - (F_1 \cap F_2)$

If $e \notin F_2$, then, as



$e \in cl(F_1 \cap F_2) \subseteq cl(F_2)$

we contradict that F_2 is a flat.

So $e \in F_2$. Similarly $e \in F_1$. But then $e \in F_1 \cap F_2$, a contradiction. We deduce that $F_1 \cap F_2$ is a flat. $\square$

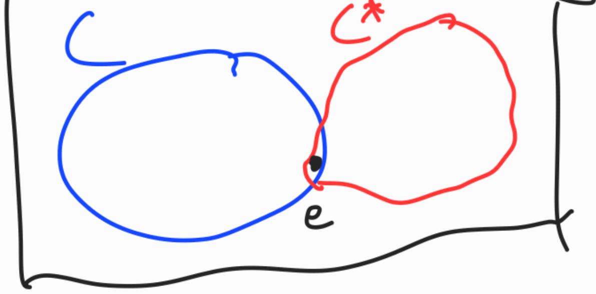
Proposition 5.12: Let M be a matroid, let C be a circuit and let C^* be a cocircuit of M. Then $|C \cap C^*| \neq 1$.

Proposition 5.12 is known as "orthogonality".

Proof: Suppose that $C \cap C^* = \{e\}$.

$E - C^*$ is a hyperplane (by Prop 3.6), and

$e \notin E - C^*$, so $e \notin cl(E - C^*)$



But e is in a circuit C, which is contained in $(E - C^*) \cup e$, so by Proposition 5.9, $e \in cl(E - C^*)$, which is a contradiction.

Hence $|C \cap C^*| \neq 1$. $\square$

Recall (C3), the "circuit elimination axiom":

(C3): If $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$ and $C_1 \neq C_2$

then there exists $C_3 \in \mathcal{C}$ such that

$C_3 \subseteq (C_1 \cup C_2) - e$

The strong circuit elimination axiom is:

(C3'): If $C_1, C_2 \in \mathcal{C}$ and $e \in C_1 \cap C_2$ and $f \in C_1 - C_2$

then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$

and $f \in C_3$.

Prop 5.13 Let M be a matroid. Then $\mathcal{C}(M)$ satisfy (C3').