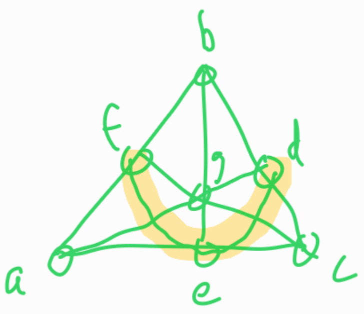
MATH432 | Lecture 16

Assignment 3 due tomorrow 5pm

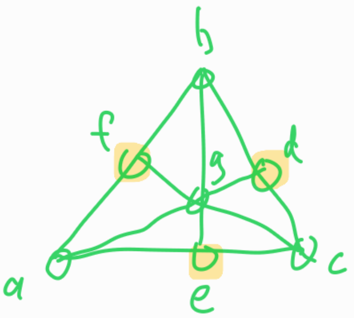
Last time: constructing a representation

We saw that F_7 is $\mathbb{F}$-representable iff $\mathbb{F}$ has characteristic 2

F_7^- iff $\mathbb{F}$ has characteristic not 2.



F_7



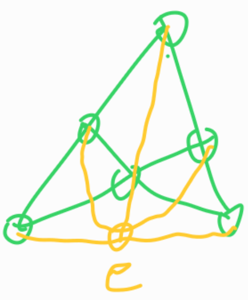
F_7^-

Side note:

$\{d, e, f\}$ is a circuit and a hyperplane in F_7.

Whenever M is a matroid with a circuit-hyperplane X, there is a matroid M' on ground set $E(M)$ whose bases are $\mathcal{B}(M) \cup \{X\}$. See Prop 5.15 in online notes.

Exercise: Which fields is the matroid M representable over?



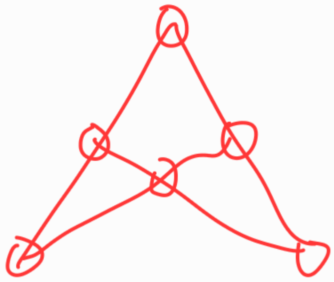
M

$M = F_7^- \setminus e \Rightarrow M$ is representable over fields of characteristic not 2

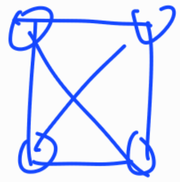
$M = F_7 \setminus e \Rightarrow M$ is representable over fields of characteristic 2

Thus M is regular (representable over all fields).

Alternatively:



$M \cong M(K_4)$

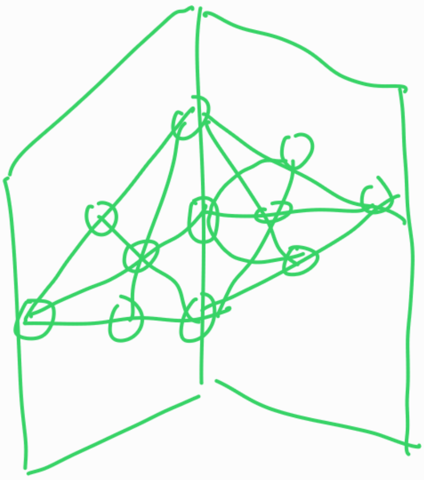


K_4

Since M is graphic, it is regular (we'll prove this soon).

Exercise:

What fields is M representable over?

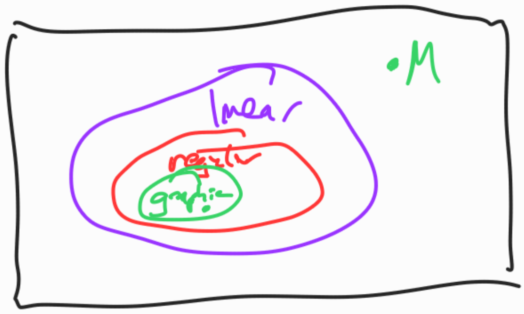


M

Since M has a F_7^--minor it is not representable over fields of characteristic not 2

Since M has a F_7-minor it is not representable over fields of characteristic 2

So M is not representable over any field.



Recall from last time, by scaling rows and columns, we went from:

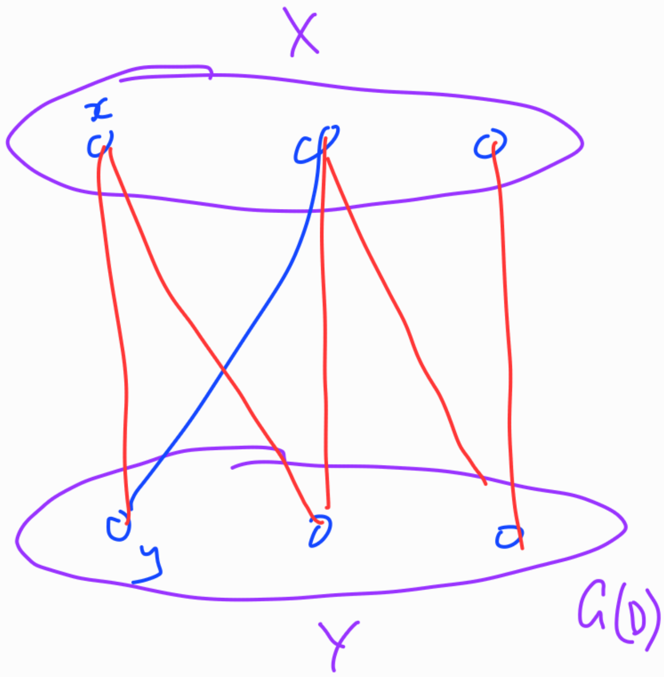
$$\left[\begin{array}{c|cccc} & 0 & * & * & * \\ I & * & 0 & * & * \\ & * & * & 0 & * \end{array}\right] \longrightarrow \left[\begin{array}{c|cccc} & 0 & 1 & 1 & 1 \\ I & 1 & 0 & 1 & * \\ & * & * & 0 & 1 \end{array}\right]$$

Which entries can be scaled to 1?

If we have

$$\begin{bmatrix} \overbrace{I_r}^{X} & \overbrace{D}^{Y} \end{bmatrix} \quad \text{or} \quad X\left\{ \begin{array}{c} \\ x \\ \end{array} \right. \overbrace{\begin{bmatrix} & & y \\ & D & \\ & & \boxed{*} \end{bmatrix}}^{Y}$$

and draw a bipartite graph $G(D)$ such that xy is an edge precisely when $D_{xy} \neq 0$, then we can choose any forest of $G(D)$ and scale the corresponding entries to any non-zero entries (for example, all 1s). See Proposition 7.2.



Projective geometries and matroid representations

For a finite field $\mathbb{F}$, we could construct a matrix A consisting of all distinct vectors in $\mathbb{F}^r$.

What then is $M[A]$?

Let $\mathbb{F}$ be a finite field, so $\mathbb{F} = GF(q)$

for some prime power q.

Consider the r-dimensional vector space $\mathbb{F}^r$

for some non-negative integer r.

We view $\mathbb{F}^r$ as a matroid with q^r elements,

which we denote $V(r, q)$.

The matroid has a loop corresponding to $\underset{\sim}{0}$,

and parallel classes consisting of $q-1$ elements.

e.g. for $GF(5)$, $\{\underset{\sim}{v}, 2\underset{\sim}{v}, 3\underset{\sim}{v}, 4\underset{\sim}{v}\}$

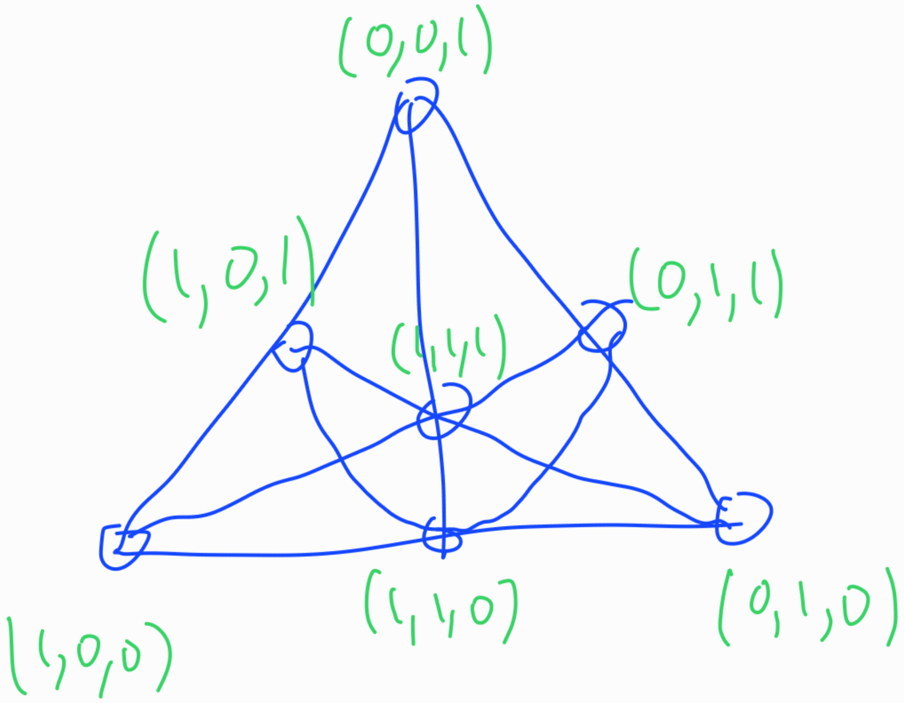
is a parallel class when $v \neq 0$

$PG(r-1, q)$ is the $(r-1)$-dimensional vector space over $GF(q)$ obtained by removing the $\underset{\sim}{0}$ vector and all but one element of each parallel class — it corresponds to the simple matroid associated with $V(r,q)$.

It has $\dfrac{q^r - 1}{q-1}$ elements.

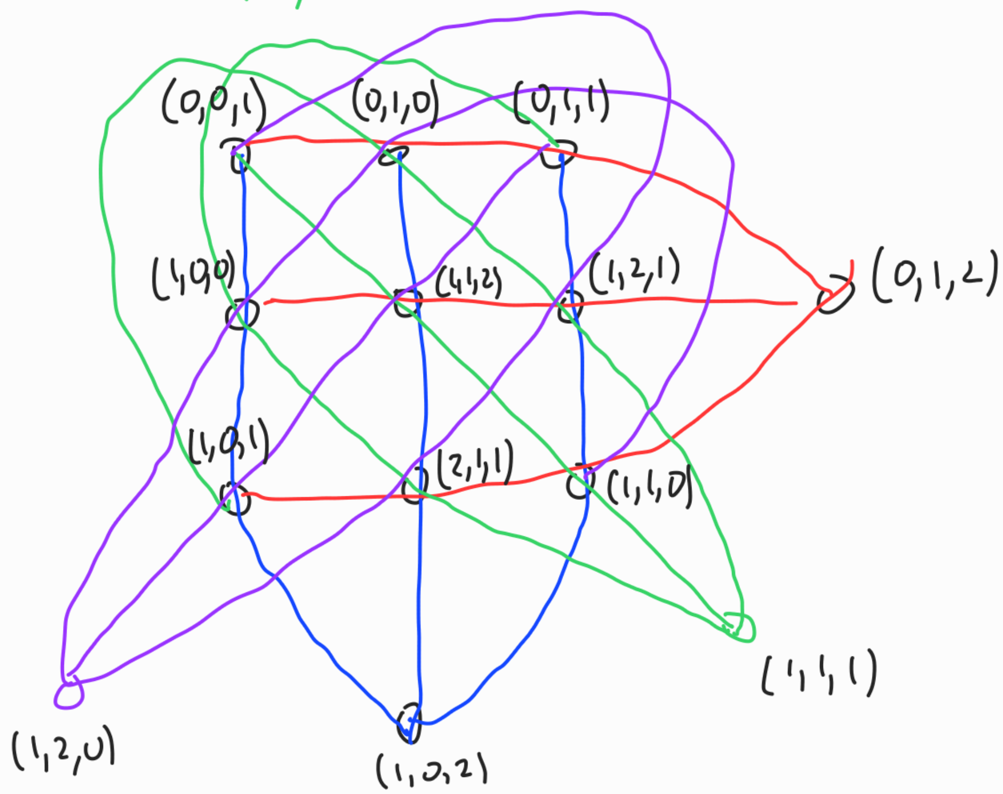
e.g. $PG(2,2)$ has $\dfrac{2^3-1}{2-1} = 7$ elements.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$



$PG(2,2) \cong F_7$

PG(2,3) has 13 elements



$PG(r-1,q)$ is called a <u>projective geometry</u>.

Projective geometries are to $\mathbb{F}$-representable matroids as complete graphs are to graphs.

<u>Proposition</u>: A simple rank-r matroid M is representable over $GF(q)$ if and only if M is isomorphic to a restriction of $PG(r-1,q)$.

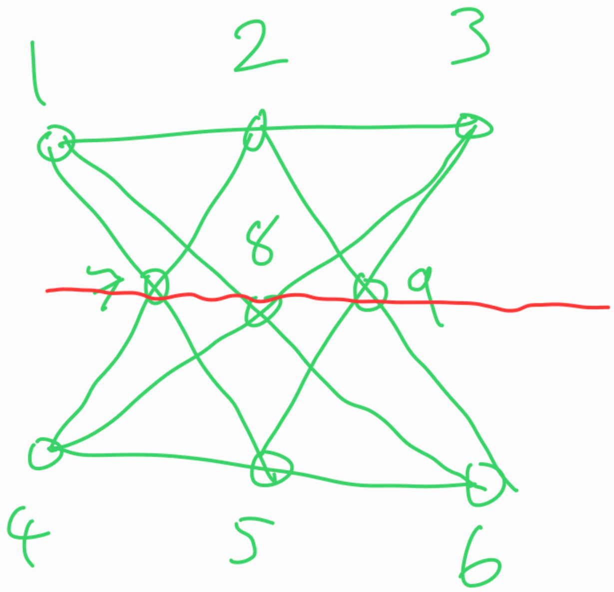
Finding a representation for a matroid is like finding homogeneous coordinates for the points in space.

(These are coordinates used in projective geometry

* non-zero scalar multiples represent the same point

* there is no $\underset{\sim}{0}$ vector.)

Next, another example that is not representable over any field.



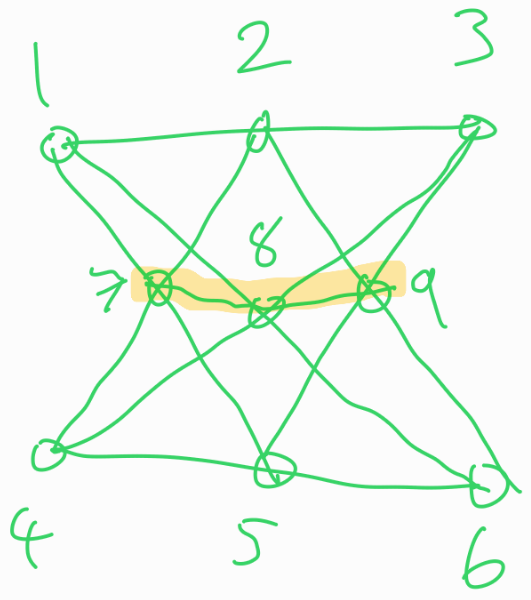
Pappus's hexagon theorem is a geometric result that says that if $\{1,2,3\}$ and $\{4,5,6\}$ are collinear, and the points of intersection of the lines

$$\left.\begin{matrix} 15, 24 \\ 16, 34 \\ 35, 26 \end{matrix}\right\} \text{ are } \left\{\begin{matrix} 7 \\ 8 \\ 9 \end{matrix}\right.$$

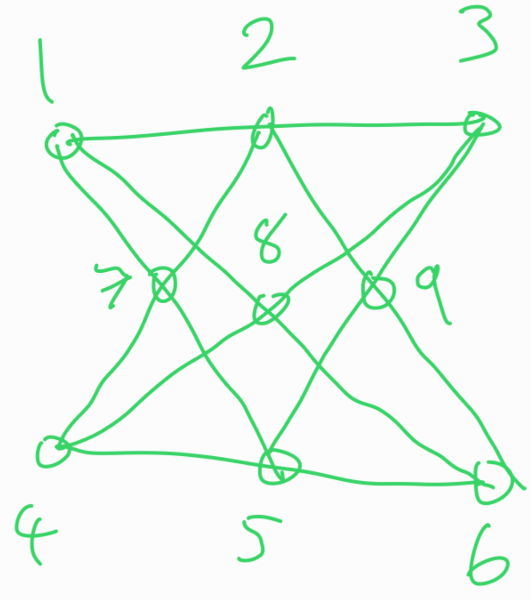
respectively

then $\{7,8,9\}$ are collinear.

It holds for any projective geometry $PG(r-1,q)$.



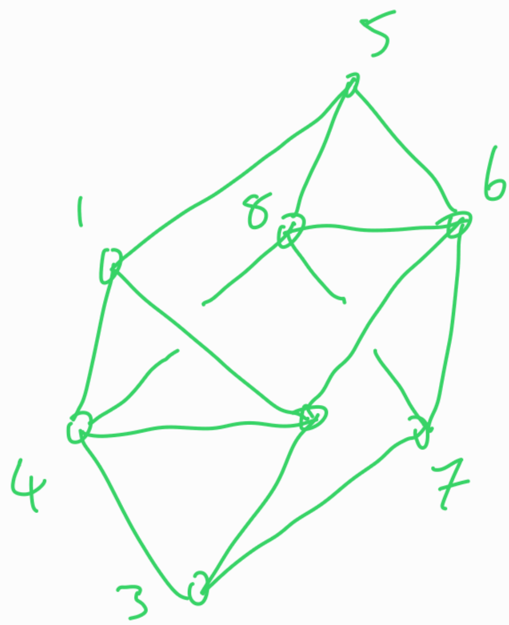
relax the circuit-hyperplane $\{7,8,9\}$.



Pappus matroid. Non-Pappus matroid.

It follows from the previous proposition and Pappus's hexagon theorem that the non-Pappus matroid is not representable over any finite field — it turns out that it is not representable over any infinite field either.

Consider the rank-4 sparse paving matroid on ground set $\{1,2,\dots,8\}$



with non-spanning circuits

$\{1,5,2,6\}$, $\{1,5,4,8\}$, $\{2,6,3,7\}$, $\{3,7,4,8\}$ $\{2,6,4,8\}$.

Note that $\{1,5,3,7\}$ is a basis.

This matroid is called the Vámos matroid.

Exercise: Show the Vámos matroid is not representable over any field (by trying to construct a representation as in the previous lecture.)