MATH432 | Lecture 17

Recall that a matroid is <u>regular</u> if it is representable over every field.

We'll prove...

Theorem 7.10: Every graphic matroid is regular.

The <u>vertex-edge incidence matrix</u> of a graph G is a matrix A with rows labelled by $V(G)$ and columns labelled by $E(G)$, where

$$\begin{array}{cc} & E(G) \\ & e \\ V(G) \quad v & \begin{bmatrix} & & \\ & A & \square \\ & & \end{bmatrix} \end{array}$$

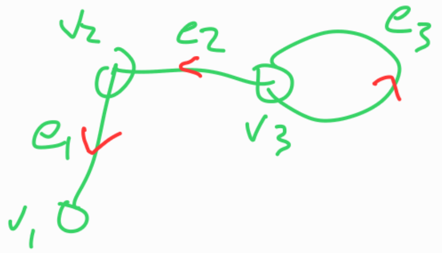
$$A_{ve} = \begin{cases} 1 & \text{if } v \text{ is incident with } e \\ 0 & \text{otherwise} \end{cases}$$

An <u>oriented vertex-edge incidence matrix</u> can be obtained from the vertex-edge incidence matrix by:

* for each non-loop edge (with two 1's in the column) replace one $+1$ with a -1.
* for each loop, all entries in the column are 0.

e.g.

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$



$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{matrix} v_1 \\ v_2 \\ v_3 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix} \end{array}$$

Theorem 7.10: Every graphic matroid is regular.

Proof: Let G be a graph and let $\mathbb{F}$ be a field. We let A be an oriented vertex-edge incidence matrix for G, where we view A as a matrix over $\mathbb{F}$ (note that if $\mathbb{F} = GF(2)$, then $-1 = 1$) where the columns are labelled by edges of G.

We claim that $M[A] = M(G)$.

It suffices to show that each circuit of $M(G)$ contains a circuit of $M[A]$ and each circuit of $M[A]$ contains a circuit of $M(G)$ — then the circuits coincide by (C2), so $M[A] = M(G)$.

Let C be a circuit of $M(G)$, so C is the edge set of a cycle of G. If $C = \{e\}$ is a loop, then the e column of A is $\underset{\sim}{0}$, so $\{e\}$ is dependent in $M[A]$. So assume $|C| \geq 2$.

Let $C = \{e_0, e_1, e_2, \dots, e_{t-1}\}$ where $v_0, e_0, v_1, e_1, \dots, v_{t-1}, e_{t-1}, v_0$ is a walk in G.

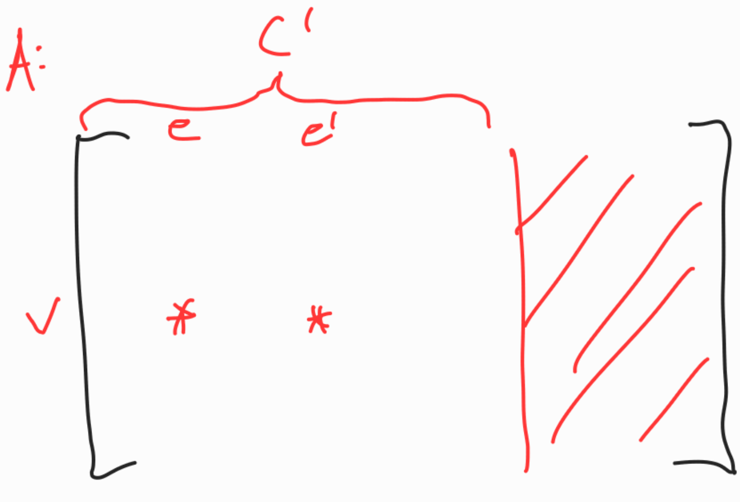
Up to swapping rows and columns, and scaling columns, we may assume that A, restricted to the columns $\{e_0, e_1, \dots, e_{t-1}\}$ is of the form:

$$\begin{array}{c|c}
 & \begin{matrix} e_0 & e_1 & e_2 & \cdots & e_{t-1} \end{matrix} \\
\begin{matrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{t-1} \\ \\ \\ \end{matrix} &
\left[\begin{matrix}
1 & 0 & 0 & 0 & & -1 \\
-1 & 1 & 0 & \vdots & & \\
0 & -1 & 1 & 0 & & \\
\vdots & 0 & -1 & 1 & & \\
\vdots & \vdots & 0 & -1 & \ddots & \\
\vdots & \vdots & \vdots & 0 & & \\
0 & 0 & 0 & 0 & & 1 \\
\hline
 & & & 0 & & \\
\end{matrix}\right]
\end{array}$$

As the columns sum to $\underset{\sim}{0}$, $C = \{e_0, e_1, \dots, e_{t-1}\}$ is dependent in $M[A]$.

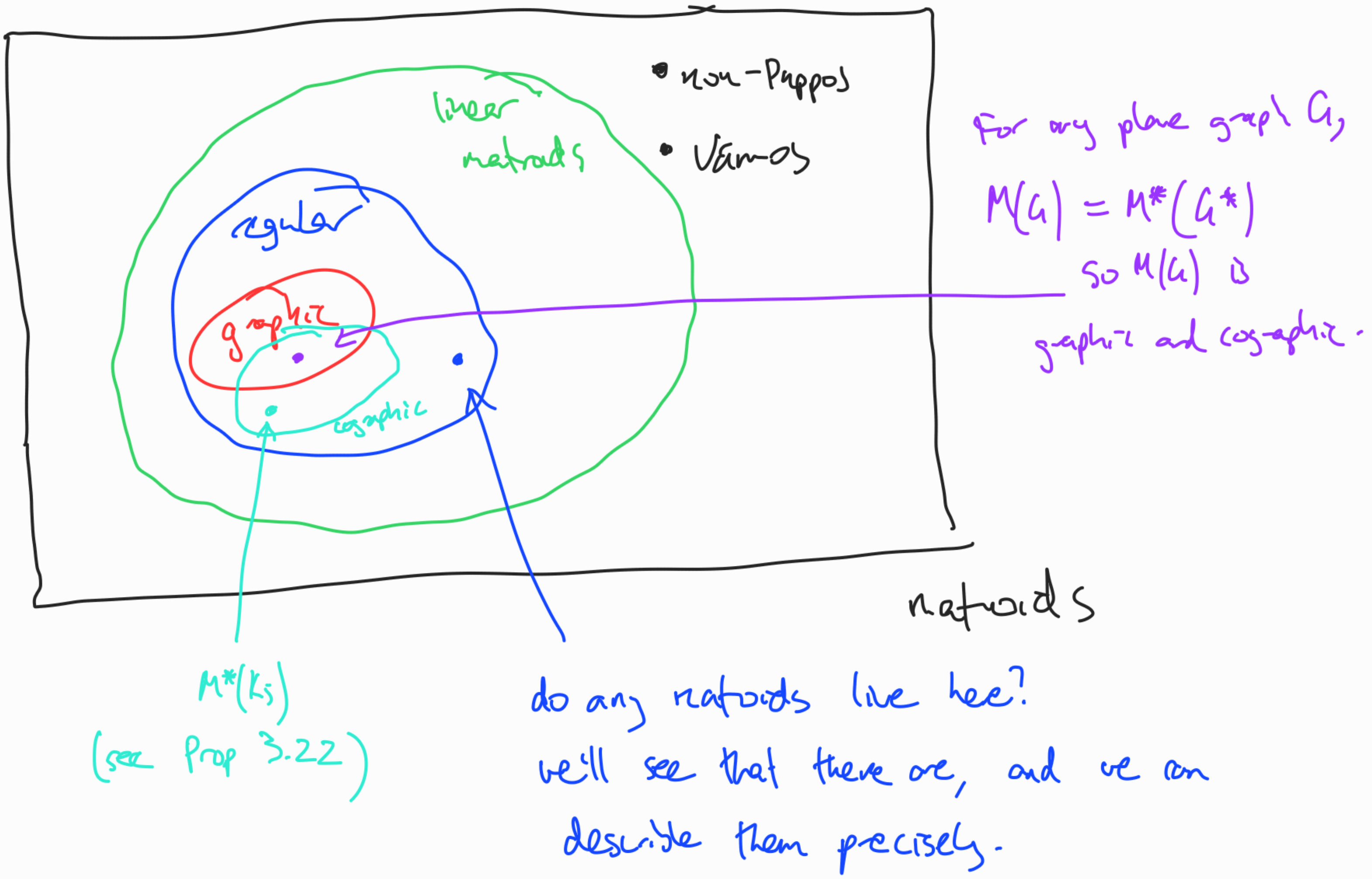
Now let C' be a circuit in $M[A]$. Then the columns of a corresponding to C' form a minimal linearly dependent set. If $C' = \{e\}$ is a singleton, then e is a loop in G. So assume otherwise.

If v is a row such that $A_{ve} \neq 0$ for some column $e \in C'$, then it is not the only such column in C' (otherwise C' is not a minimal linearly dependent set),



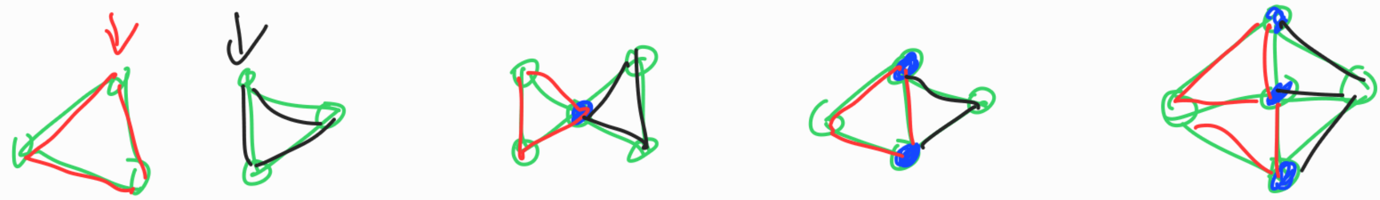
So there is some $e' \in C' - e$ such that $A_{ve'} \neq 0$.

Hence, if a vertex is incident with an edge in C', then it is incident with at least two edges in C'. Now the edge-induced subgraph of G on C' has minimum degree at least 2, so it contains a cycle (since a forest has at least one vertex of degree at most 1). This shows C' contains a circuit in $M(G)$, as required. □



Connectivity   Firstly in graphs.

e.g.



not connected   connected, not 2-connected   2-connected but not 3-connected   3-connected but not 4-connected

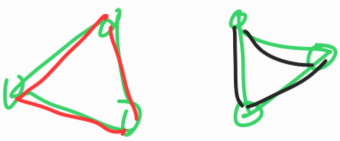
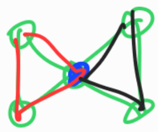
Def^n: For a graph G, we say a bipartition (E_1, E_2) of $E(G)$ is a *k-separation* if $|V(G[E_1])| > k$ and $|V(G[E_2])| > k$

and $|V(G[E_1]) \cap V(G[E_2])| \le k$.

Def^n: A graph G is *k-connected* if $|V(G)| > k$ and G has no j-separations for $j < k$.

Now we want to generalise this to matroids.

First, a couple of observations:

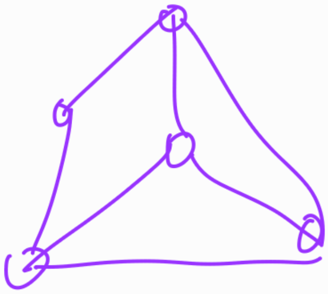
(1) G_1  $\not\cong$  G_2

but $M(G_1) = M(G_2)$

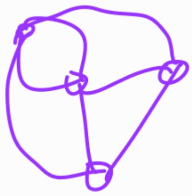
so we can't distinguish between "1-connected" and "not 1-connected".

2

G



G^*



G is not 3-connected

but G^* is 3-connected

(for matroids, we would like

M is k-connected $\iff$ M^* is k-connected)

Connectivity in matroids

Let M be a matroid on ground set E.

We define the connectivity function of M to be the function

$$\lambda_M : 2^E \longrightarrow \mathbb{N} \quad \text{for which}$$

$$\lambda_M(X) = r(X) + r(E-X) - r(M)$$

for $X \subseteq E$.

Definition: For a partition $(X, E-X)$ of E, we say $(X, E-X)$ is a k-separation if $|X| \geq k$ and $|E-X| \geq k$ and $\lambda_M(X) < k$.

A matroid M is n-connected if M has no k-separations for $k < n$.

Definition: For a partition $(X, E-X)$ of E, we say $(X, E-X)$ is a vertical k-separation if $r(X) \geq k$ and $r(E-X) \geq k$ and $\lambda_M(X) < k$.

A matroid M is vertically n-connected if M has no vertical k-separations for $k < n$.

We'll see $M(G)$ is vertically n-connected $\iff$ G is n-connected.

On the other hand

M is n-connected $\iff$ M^* is n-connected.