MATH4432 | Lecture 2

Recap: We considered points in $\mathbb{R}^n$, and defined when a set of points is a <u>basis</u>

<u>independent set</u>

<u>dependent set</u>

We defined a <u>matroid</u> to be:

a pair $(E, \mathcal{B})$ such that

(B1) $\mathcal{B} \neq \emptyset$

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 - B_2$ then there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$

We get the family of independent sets by closing $\mathcal{B}$ under subsets

i.e. $\mathcal{I} = \{ I \subseteq B : B \in \mathcal{B} \}$.

e.g. Consider $U_{3,5}$ on ground set $\{a, b, c, d, e\}$

$$U_{3,5} = (\{a,b,c,d,e\}, \{B \subseteq \{a,b,c,d,e\} : |B| = 3\})$$

Geometric rep^n:

5 points on a plane

e.g. For $U_{4,4}$ 

For $U_{4,6}$: 

note 6 points freely placed in 3 dimensions!

Isomorphism: Let $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$ be matroids.

M_1 is *equal* to M_2, denoted $M_1 = M_2$, if $E_1 = E_2$ and $\mathcal{B}_1 = \mathcal{B}_2$.

M_1 is *isomorphic* to M_2, denoted $M_1 \cong M_2$, if there exists a bijection $\varphi : E_1 \to E_2$ such that for all $X \subseteq E_1$, $X \in \mathcal{B}_1$ if and only if $\varphi(X) \in \mathcal{B}_2$.

Here we're using the notation $\varphi(X) = \{\varphi(x) : x \in X\}$.

Def^n: The *rank* of a matroid M is the size of a basis in M.

e.g. $U_{3,5}$ has rank 3

$U_{r,n}$ has rank r.

A geometric representation of a matroid with rank r will be in $(r-1)$-dimensional space.

Lemma A: Let $M = (E, \mathcal{B})$ be a matroid with independent sets $\mathcal{I}$. Then $\mathcal{I}$ satisfies (I1) − (I3).

(I1) $\emptyset \in \mathcal{I}$

(I2) If $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ then $I_2 \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$, then $\exists e \in I_1 - I_2$ s.t. $I_2 \cup e \in \mathcal{I}$.

Proof: Recall that $\mathcal{I} = \{ I \subseteq B : B \in \mathcal{B} \}$. (*)

Since $\mathcal{B} \neq \emptyset$ by (B1), there exists some $B \in \mathcal{B}$ and $\emptyset \subseteq B$, so $\emptyset \in \mathcal{I}$ by (*), so (I1) holds.

Next, suppose $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$. Then $I_1 \subseteq B$ for some $B \in \mathcal{B}$ (by (*)). So $I_2 \subseteq I_1 \subseteq B$, hence $I_2 \in \mathcal{I}$ (by (*)). Hence (I2) holds.

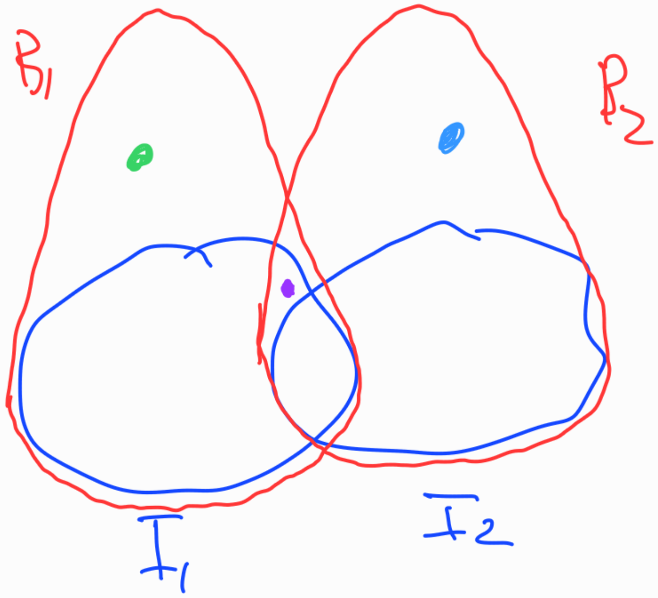
Suppose (I3) doesn't hold. Then there exist $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$ such that for every $e \in I_1 - I_2$ we have that $I_2 \cup e \notin \mathcal{I}$.

By (*), there exist some $B_1, B_2 \in \mathcal{B}$ such that $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$. We choose B_1 and B_2 so that $|B_1 \cap B_2|$ is maximized.

If there exists $x \in (I_1 \cap B_2) - I_2$,

then $x \in I_1 - I_2$ and $I_2 \cup x \subseteq B_2$, so $I_2 \cup x \in \mathcal{I}$.

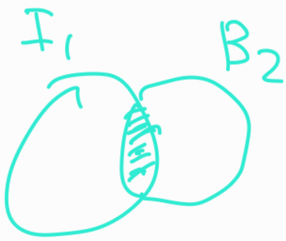
Hence no such x exists.



Thus $I_1 \cap B_2 \subseteq I_2$ so

$$
\begin{aligned}
I_1 - I_2 &\subseteq I_1 - (I_1 \cap B_2) \\
&= I_1 - B_2 \\
&\subseteq I_1 - I_2
\end{aligned}
$$

since $I_2 \subseteq B_2$



So $I_1 - I_2 = I_1 - B_2$.

Next, if there exists $x \in B_1 - (I_1 \cup B_2)$, then $x \in B_1 - B_2$. By (B2), there exists $y \in B_2 - B_1$ such that $(B_1 - \{x\}) \cup \{y\} \in \mathcal{B}$. But $(B_1 - \{x\}) \cup \{y\}$ contains I_1 and intersects B_2 in more elements than B_1 does, contradicting our choice of B_1 and B_2.

So no such x exists.

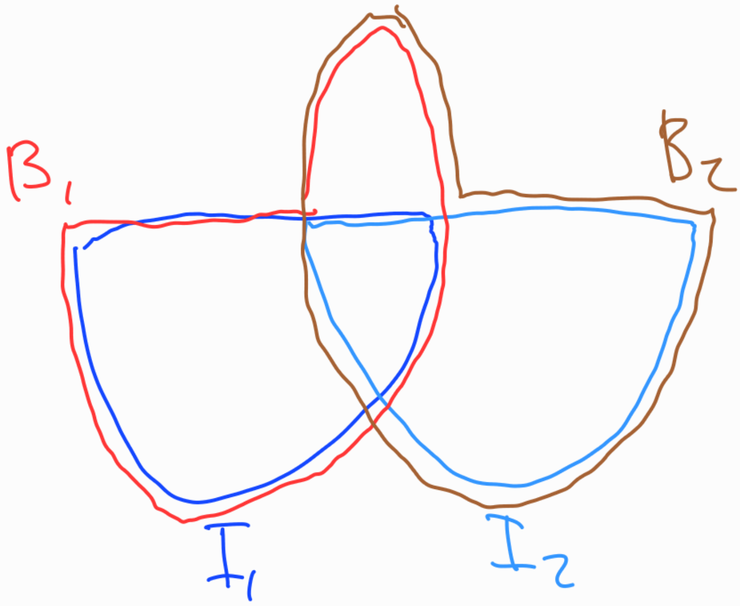
Now $B_1 - (I_1 \cup B_2) = \emptyset$,

so $\quad B_1 - B_2 \subseteq I_1 - B_2$

$$= I_1 - I_2$$
$$\subseteq B_1 - B_2.$$

Hence $B_1 - B_2 = I_1 - I_2$.

By symmetry, there is no $x \in B_2 - (I_2 \cup B_1)$,
implying $B_2 - B_1 = I_2 - I_1$.



By Prop 1.3 $|B_1| = |B_2|$

So $|B_1 - B_2| = |B_2 - B_1|$

Hence
$$\begin{aligned} |I_1 - I_2| &= |B_1 - B_2| \\ &= |B_2 - B_1| \\ &= |I_2 - I_1| \end{aligned}$$

Therefore
$$\begin{aligned} |I_1| &= |I_1 \cap I_2| + |I_1 - I_2| \\ &= |I_1 \cap I_2| + |I_2 - I_1| = |I_2| \end{aligned}$$

This is a contradiction, as $|I_2| < |I_1|$.

Hence (I3) is satisfied. $\square$

Def^n: A circuit in a matroid is a minimal dependent set.

For a matroid M, we write $E(M)$ to denote the ground set of M

$\mathcal{I}(M)$ to denote the independent sets

$\mathcal{B}(M)$ bases

$\mathcal{C}(M)$ circuits.

e.g. $\mathcal{C}(U_{r,n}) = \{ C \subseteq E(U_{r,n}) : |C| = r+1 \}$.

Defn: A matroid M is uniform if there exists r, n such that $M \cong U_{r,n}$.

Exercise: Let M be a matroid of rank r.

Prove that: M is uniform if and only if every circuit of M has size $r+1$.

For a matroid with the following geometric rep^n:

$\{a, b\}$ and $\{c, d, e\}$ are circuits.

Thm: Let E be a set and let $\mathcal{C}$ be a family of subsets of E satisfying the following properties:

(C1) $\emptyset \notin \mathcal{C}$

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - \{e\}$.

C3 is the circuit elimination property

Then there is a matroid M with ground set E whose family of circuits is $\mathcal{C}$,

and $\mathcal{I}(M) = \{X \subseteq E : C \nsubseteq X \text{ for each } C \in \mathcal{C}\}$.

Conversely, if M is a matroid, then the circuits of M, $\mathcal{C}(M)$, satisfy (C1), (C2) and (C3).

For a matroid M, we use $r(M)$ to denote the rank of M.

e.g. $r(U_{r,n}) = r$.

Def^n: Let M be a matroid, let $X \subseteq E(M)$.

The rank of X is the maximum size of an independent set contained in X.

The rank function of M is the function $r_M : 2^E \to \mathbb{N}$ that given a subset $X \subseteq E(M)$, outputs the rank of X.

$$r(M) = r_M(E(M))$$

rank of a matroid — rank of a set in a matroid.

Def^n: A matroid of rank r is <u>paving</u> if every circuit has size at least r.

Exercise: Let E be a set, let r be an integer $0 < r < |E|$. Let $\mathcal{C}'$ be a collection of r-element subsets of E such that

$$|C_1 \cap C_2| < r-1 \quad \text{for all distinct } C_1, C_2 \in \mathcal{C}'.$$

Let $\mathcal{B} = \{B \subseteq E : |B| = r \text{ and } B \notin \mathcal{C}'\}$.

Prove that $(E, \mathcal{B})$ is a matroid.