MATH432 | Lecture 6

Assignment 1 due Friday 5pm

Recap: last time, we saw that for any graph G, there is a matroid $M(G)$.

What is the rank of $M(G)$?

In other words: what is the size (number of edges) of the longest forest in G?

If G is connected and has n vertices, then a spanning tree has $n-1$ edges.

If G is connected, $r(M(G)) = |V(G)| - 1$.

Lemma: Let G be a graph with n vertices and c components, and let F be a maximal forest in G. Then $|F| = n-c$.

In general: $r(M(G)) = |V(G)| - c(G)$

where $c(G)$ is the number of components of G.

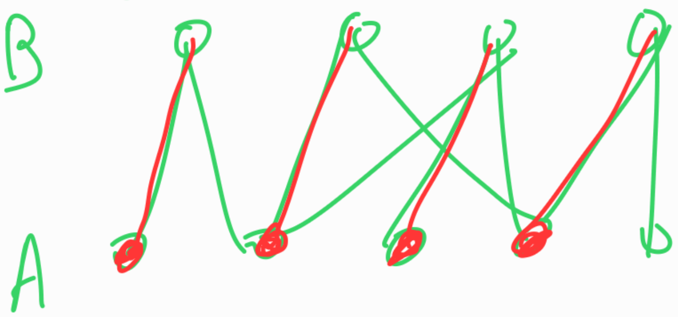
Transversal matroids These capture the notion of independence of a set of matching edges in a bipartite graph.

A graph G is bipartite if there are disjoint sets A, B such that $V(G) = A \cup B$ and every edge has one end in A and one end in B.

(A, B) is the bipartition of G.

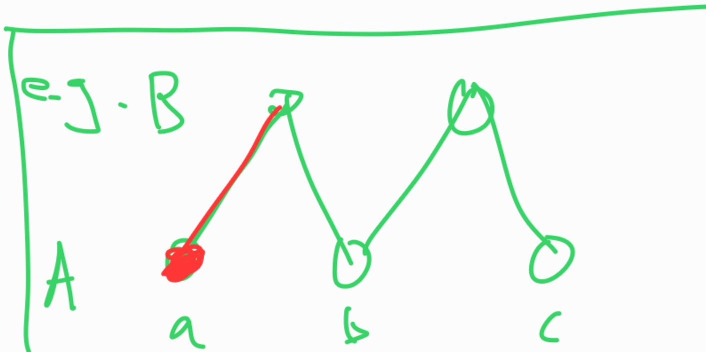
A matching in a graph is a set of pairwise non-adjacent edges

e.g.



For a bipartite graph with bipartition (A, B), then for $X \subseteq A$, we say X is matchable if there is a matching M such that, for each $x \in X$ x is incident to an edge in M.

Theorem 2.17: let G be a bipartite graph with bipartition



(E, A). Let $\mathcal{I}$ be the matchable subsets of E. Then $\mathcal{I}$ is the family of independent sets of a matroid on ground set E.

$\{a,b\}$
$\{a,c\}$
$\{b,c\}$ is matchable

$\{a,b,c\}$ is not matchable.

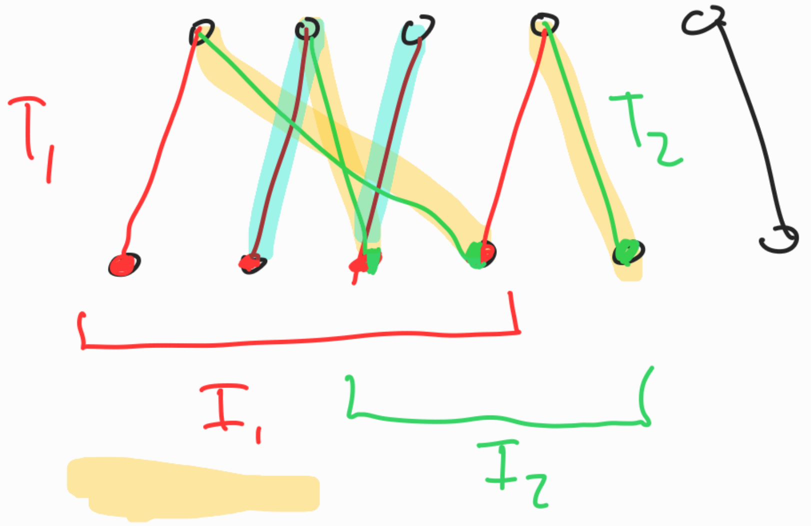
<u>Proof:</u> We'll show (I1)-(I3) hold.

The empty set is matchable so (I1) holds.

If $X \subseteq E$ is matchable, then a matching that certifies this also certifies that any $X' \subseteq X$ is matchable, so (I2) holds.

It remains to show (I3) holds. Let $I_1, I_2 \in \mathcal{I}$ with $|I_2| < |I_1|$. Then I_1 and I_2 matchable subsets of E.

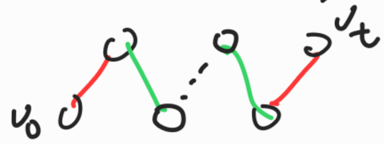
Let T_1 and T_2 be matchings that certify that I_1 and I_2 are matchable, respectively.



By removing redundant edges, we may assume that $|T_1| = |I_1|$ and $|T_2| = |I_2|$. Consider $G[T_1 \cup T_2]$.

Every vertex in this graph has degree at most 2. Hence every component of $G[T_1 \cup T_2]$ is either a path or a cycle. Pick such a component with fewer edges in T_2 than T_1 (such a component exists since $|T_2| = |I_2| < |I_1| = |T_1|$)

Then this component is a path



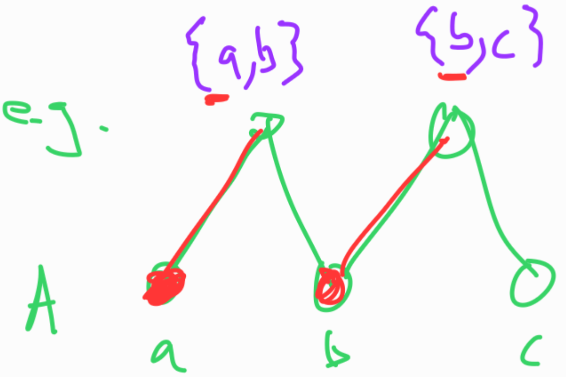
$$v_0, e_1, v_1, e_2, v_2, e_3, \ldots, v_{t-1}, e_t, v_t.$$

with $v_0 \in I_1 - I_2$ and $e_1, e_t \in T_1$.

Let $T' = (T_2 - \{e_2, e_4, \ldots, e_{t-1}\}) \cup \{e_1, e_3, \ldots, e_t\}$.

Then T' is a matching and this matching certifies that $I_2 \cup \{v_0\}$ is matchable. So (I3) holds. $\square$

e.g.



$\{a,b\}$
$\{a,c\}$ is matchable
$\{b,c\}$

$\{a,b,c\}$ is not matchable.

$E = \{a, b, c\}$

$\mathcal{B} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$

$(E, \mathcal{B}) \cong U_{2,3}$.

Note: we can view a bipartite graph as a family of subsets of a set, e.g. for the above example, we have

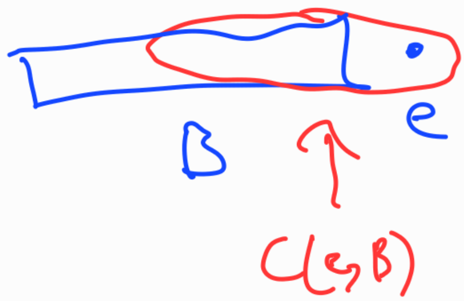
$\mathcal{A} = \{\{a, b\}, \{b, c\}\}$. Then a matchable set is a transversal of $\mathcal{A}$ or a system of distinct representatives.

Let $\mathcal{A}$ be a family of subsets of a set E. Let M be the matroid described in the last theorem. We denote this matroid as $M[\mathcal{A}]$, and say that $\mathcal{A}$ is a presentation of M.

When a matroid $M \cong M[\mathcal{A}]$ for some $\mathcal{A}$, we say M is <u>transversal</u>.

<u>Duality</u> Matroid duality generalises the notions of duality for planar graphs, and orthogonality in vector spaces.

Let M be a matroid. Let B be a basis of M and $e \in E(M) - B$. Recall that there is a unique circuit contained in $B \cup e$, which we called the fundamental circuit of e with respect to B, and denoted $C(e, B)$.



<u>Lemma</u> 3.2: Let $M = (E, \mathcal{B})$ be a matroid. Let

$$\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}.$$

Then there is a matroid on ground set E, whose family of bases is $\mathcal{B}^*$.

We'll call $M^* = (E, \mathcal{B}^*)$ the <u>dual</u> of M.

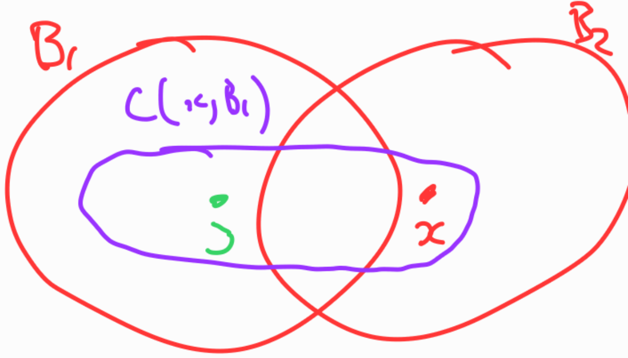
Consider the following property for a set E and family of subsets $\mathcal{B}$.

(B2') If $B_1, B_2 \in \mathcal{B}$ and $x \in B_2 - B_1$, then there exists $y \in B_1 - B_2$ such that $(B_1 - \{y\}) \cup \{x\} \in \mathcal{B}$.

<u>Proposition 3.1</u>: Let $\mathcal{B}$ be the family of bases of a matroid. Then (B2') holds.

<u>Proof</u>: Let $B_1, B_2 \in \mathcal{B}$ and $x \in B_2 - B_1$. To show (B2') holds, we want to show there is some $y \in B_1 - B_2$ s.t. $(B_1 - \{y\}) \cup \{x\} \in \mathcal{B}$.

There is a unique circuit $C(x, B_1)$ contained in $B_1 \cup \{x\}$ where $x \in C(x, B_1)$ (by Prop 1.12). Let $y \in C(x, B_1) - \{x\}$.



Then, since $C(x, B_1)$ is the unique circuit contained in $B_1 \cup \{x\}$, the set $(B_1 - \{y\}) \cup \{x\}$ does not contain any circuits, so it is independent. As $|(B_1 - \{y\}) \cup \{x\}| = |B_1|$, we have $(B_1 - \{y\}) \cup \{x\} \in \mathcal{B}$ by Proposition 1.3. $\square$

We'll use this to prove Lemma 3.2 next time.