MATH432 | Lecture 8

Recap: duality

For any matroid $M = (E, \mathcal{B})$, letting $\mathcal{B}^* = \{E - B : B \in \mathcal{B}\}$

$M^* = (E, \mathcal{B}^*)$ is also a matroid, called the dual of M.

For $X \subseteq E(M)$,

X is codependent in M $\iff$ $E - X$ is spanning in M

X is a cocircuit in M $\iff$ $E - X$ is a hyperplane in M

$r^*(X) = |X| + r(E - X) - r(M)$.

Example: Let r and n be non-negative integers with $r \leq n$.

Is $U_{r,n}^*$ uniform?

Since $\mathcal{B}(U_{r,n}) = \{X \subseteq E(U_{r,n}) : |X| = r\}$

$\mathcal{B}(U_{r,n}^*) = \{X \subseteq E(U_{r,n}) : |X| = n - r\}$

that is $U_{r,n}^* = U_{n-r,n}$.

For a class of matroids $\mathcal{M}$, we say $\mathcal{M}$ is closed under duality if for each $M \in \mathcal{M}$, we have $M^* \in \mathcal{M}$.

Uniform matroids are closed under duality.

Exercise: Construct a transversal matroid whose dual is not transversal.

Fix a field $\mathbb{F}$. Is the class of $\mathbb{F}$-representable matroids closed under duality?

$\mathbb{F}$-representable matroids Let A be a matrix over a field $\mathbb{F}$, and $M = M[A]$. In general, A does not uniquely determine M.

e.g. 1 swap 2 columns (along with their labels).

e.g. 2 $M\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = M\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$

M remains unchanged after performing any of the following operations on A:

1) interchanging two rows
2) multiplying a row by a non-zero entry in $\mathbb{F}$
3) add one row to another

elementary row operations.

4) interchanging two columns
5) multiplying a column by a non-zero entry in $\mathbb{F}$
6) adjoining or removing zero rows
7) replacing each entry with its image under some

automorphism of the field.

e.g. for $\mathbb{C}$ $\quad a+bi \mapsto a-bi$

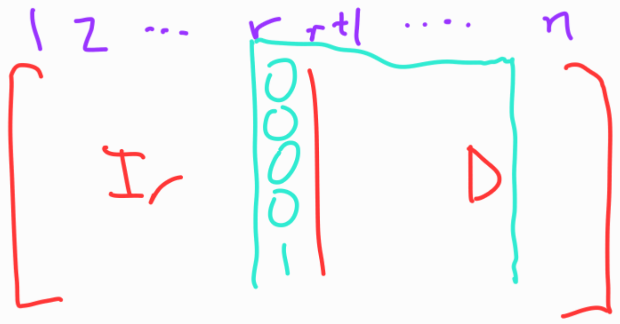
$$\left[\overbrace{\qquad\qquad}^{B}\qquad A \qquad\qquad\right] \xrightarrow{\text{row operation}} \left[\begin{array}{c|c} \overbrace{\quad I \quad}^{B} & \quad D \quad \end{array}\right]$$

By performing these operations, for any basis B of M with $r = |B|$, we can transform A into the form $[\,I_r \mid D\,]$ where I_r is the $r \times r$ identity matrix with columns labelled by B.

This representation of M is in <u>standard form</u>.

Sometimes, it is useful to just focus on the matrix D that is, to take a viewpoint where the identity is implicit and the elements of our matroid correspond to columns and rows of some matrix.

This is known as a <u>reduced representation</u>



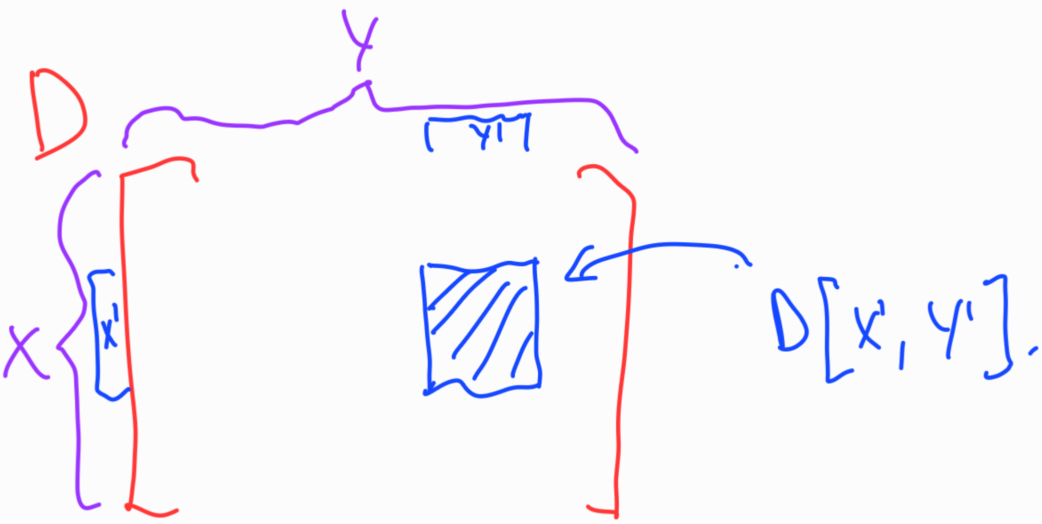
D is a reduced representation.

r columns are linearly independent

$\Leftrightarrow$ $r \times r$ submatrix has non-zero determinant

For a matrix D whose rows and columns are labelled by ordered sets X and Y respectively, and $X' \subseteq X$ and $Y' \subseteq Y$, let $D[X', Y']$ denote the submatrix of D whose rows are labelled X' and columns are labelled Y' respectively.

e.g.



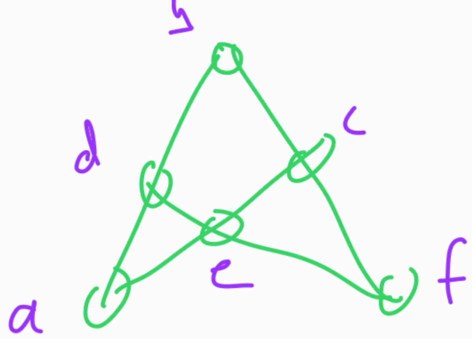
<u>Proposition</u> 3.13: Let $\mathbb{F}$ be a field and let D be a matrix over $\mathbb{F}$ with rows and columns labelled by ordered sets X and Y respectively, with $|X| = r$.

Let $\mathcal{B} = \{X\} \cup \{Z \subseteq X \cup Y : |Z| = r \text{ and } \det(D[X-Z, Y \cap Z]) \neq 0\}$

Then $\mathcal{B}$ is the family of bases of a matroid M on ground set $X \cup Y$. Moreover, $M = M[I_r \mid D]$ where the column labels of I_r and D are given by X and Y respectively.

e.g. Let
$$D = \begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{c} \begin{array}{ccc} d & e & f \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$
over $GF(2)$.

Previously, we looked at $M[I_3 \mid D]$ and saw



is a geometric representation.

We see that $\{a, c, d\}$ is a basis since
$$\det([1]) \neq 0$$

$\{b, d, e\}$ is a basis since
$$\det\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) \neq 0$$

Proof sketch of Prop 3.13:

Let $M = M[\overbrace{I_r}^{X} \mid \overbrace{D}^{Y}]$

We claim that the family of bases of M is $\mathcal{B}$.

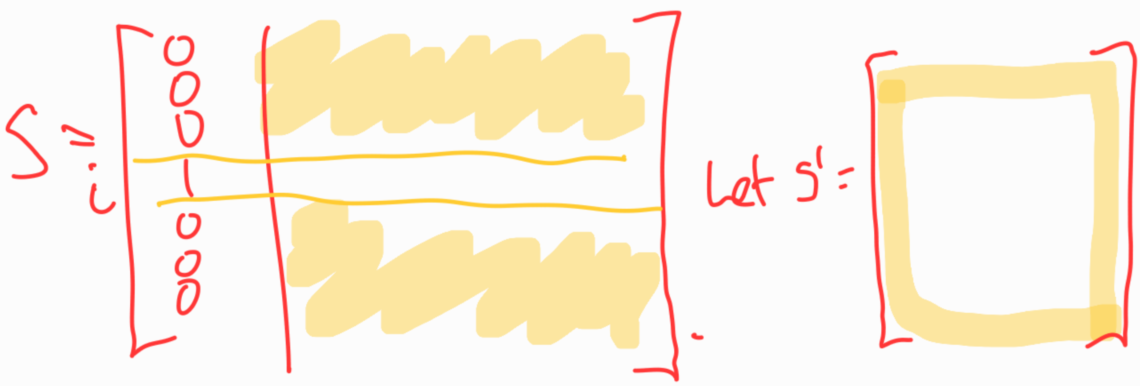
Let $Z \subseteq X \cup Y$, where $|Z| = r$.

The set Z is a basis in M

$\Leftrightarrow$ the columns labelled by Z are linearly independent

$\Leftrightarrow$ the $r \times r$ submatrix of $[I_r|D]$ consisting of columns labelled by Z has non-zero determinant

Suppose $x \in X \cap Z$, so the $r \times r$ submatrix S of $[I_r|D]$ contains the column of I labelled by x. Say the i^{th} coordinate of this column is 1 (the others are 0.



Then $|\det(S)| = |\det(S')|$

$\Leftrightarrow Z = X$ or $\det(D[X-Z, Y \cap Z]) \neq 0$.

Proposition 3.15: Let $\mathbb{F}$ be a field. Let D be an $r \times s$ matrix over $\mathbb{F}$, and let $M = M[I_r \mid D]$. Then the dual of M is

$$M[I_s \mid D^T].$$

transpose of D.

That is, if D is a reduced representation of M then D^T is a reduced rep^n of M^*.

Proof: Let X and Y be ordered sets corresponding to the rows and columns of D, respectively. So $|X| = r$ and $|Y| = s$. By Prop 3.13, the bases of M are

$$\{X\} \cup \{Z \subseteq X \cup Y : |Z| = r \text{ and } \det(D[\underbrace{X - Z}_{}, \underbrace{Y \cap Z}_{}]) \neq 0\}$$

Thus, the bases of M^* are

$Z' = (X \cup Y) - Z$

$$\{Y\} \cup \{Z' \subseteq X \cup Y : |Z'| = s \text{ and } \det(D[\underbrace{X \cap Z'}_{\text{equals}}, \underbrace{Y - Z'}_{\text{equals}}]) \neq 0\}$$

$$\left[\det(A) = \det(A^T) \Rightarrow \det(D[X', Y']) = \det(D^T[Y', X'])\right]$$

$= \{Y\} \cup \{Z' \subseteq X \cup Y : |Z'| = s \text{ and } \det(D^T[Y - Z', X \cap Z']) \neq 0\}$

which is precisely the family of bases of the matroid

$M[I_s \mid D^T]$ by Prop 3.13 $\square$

Let $[I_r \mid D]$ be a $r \times n$ matrix over $\mathbb{F}$.

It is well known that the orthogonal subspace of the row space of $[I_r \mid D]$ is the row space

of $[-D^T \mid I_{n-r}]$.

The dual of $M[I_r \mid D]$ is $M[I_{n-r} \mid D^T]$

$= M[-D^T \mid I_{n-r}]$

so duality for matroids can be viewed as an extension of the notion of orthogonality in vector spaces.