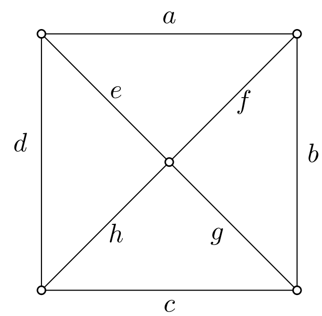
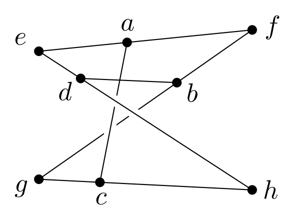
Q1. Consider the following graph G.

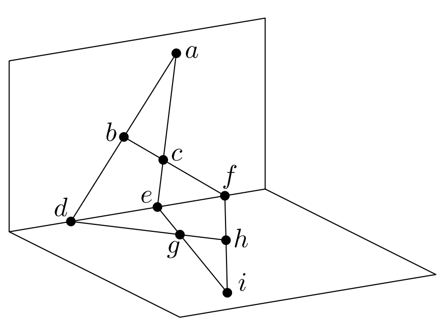


Draw a geometric representation of $M(G)$. Label the elements appropriately. **[4]**

Solution: There are different ways one can attempt to draw this. Apart from the collinearaties (i.e. $\{a, e, f\}$, $\{b, f, g\}$, $\{c, g, h\}$, $\{d, e, h\}$), you need to make clear that it is a rank-4 matroid (i.e. not all the points are lying in one plane), and that the points $\{a, b, c, d\}$ are coplanar (which one can indicate by showing that these points lie on two lines that intersect at some point).

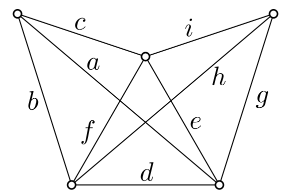


Q2. Consider the rank-4 matroid M with the geometric representation given below (and also seen in Assignment 1 Q1). Draw a graph G such that $M = M(G)$. Label the edges of G appropriately.



[4]

Solution:

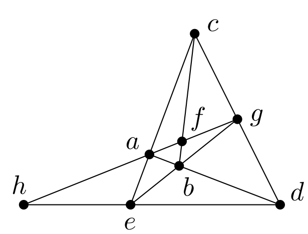


Q3. Draw a geometric representation of the ternary matroid that has a representation over GF(3) given by the following matrix. Label the elements, and provide some working.

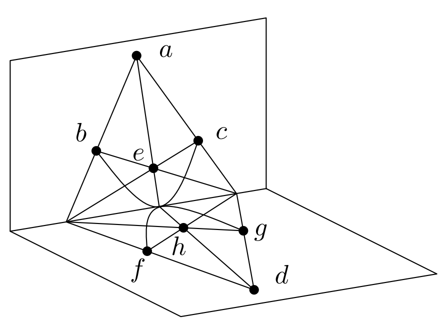
$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

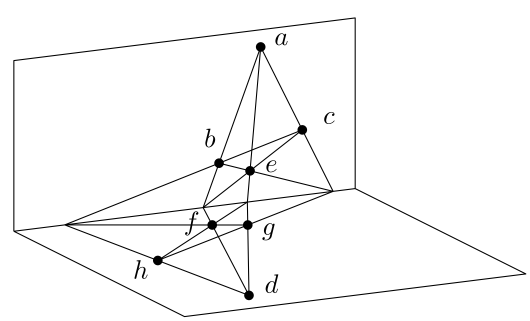
[6]

Solution: This matroid clearly has rank 3, and has no dependent sets of size at most two. The 3-element circuits are: $\{a, b, d\}, \{a, c, e\}, \{b, c, f\}, \{b, e, g\}, \{c, d, g\}, \{d, e, h\}$, and any 3-element subset of $\{a, f, g, h\}$. There are different ways to draw the geometric representation; here is one possibility:



Q4. The following diagram shows geometric representations of two rank-4 matroids. Find a matrix that represents the first matroid over the field GF(2), and a matrix that represents the second over GF(3). Label the columns appropriately, and provide some working.





[10]

Solution: Both matroids have rank 4, and $\{a, b, c, d\}$ is a basis, so (if representable) we can choose these columns to be labelled by a 4×4 identity matrix.

For this first, we can then look at the fundamental circuits relative to $\{a, b, c, d\}$, which tells us which matrix entries are zero or non-zero. Any non-zero entry must be 1, so we obtain the following GF(2)-representation:

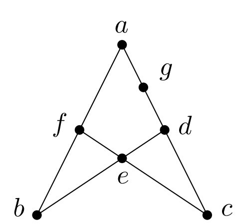
$$\begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Note that different solutions are possible (e.g., if you chose a different basis to label the identity matrix).

For the second, we start similarly and again, by considering fundamental circuits relative to the basis $\{a, b, c, d\}$, we can find whether the remaining entries are zero or non-zero. Now, however, the non-zero elements could be 1 or 2. By trial and error (we will see more methodical approaches later in the course), we can find the following GF(3)-representation:

$$\begin{array}{c} \begin{matrix} a & b & c & d & e & f & g & h \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \end{bmatrix} \end{array}.$$

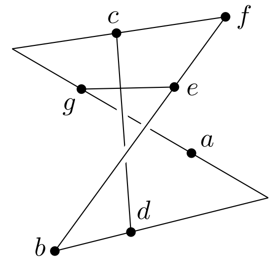
Q5. Let M be the rank-3 matroid shown below.



Draw a geometric representation of M^*. Show some working. **[6]**

Solution: Since M has rank 3 and $|E(M)| = 7$, the matroid M^* has rank 4. Since M has rank 3, hyperplanes correspond to lines of the geometric representation, and complements of these sets are cocircuits of M, i.e. circuits of M^*. Since M^* has rank 4, we're primarily interested in the circuits of M^* of size at most 4, corresponding to the lines of length at least three in M. These are: $\{a, b, f\}$, $\{c, e, f\}$, $\{b, d, e\}$, $\{a, c, d, g\}$. The complements of these sets are: $\{c, d, e, g\}$, $\{a, b, d, g\}$, $\{a, c, f, g\}$, $\{b, e, f\}$ (so these are non-spanning circuits in M).

Again, there are different ways you can draw the rank-4 matroid M^*; the important thing is to ensure that $\{b, e, f\}$ are collinear, and each of $\{c, d, e, g\}$, $\{a, b, d, g\}$, and $\{a, c, f, g\}$ are coplanar. One option is the following:



Q6. Let M be a matroid, and let e and f be elements of $E(M)$ that are not coloops. Prove that $\{e, f\}$ is a cocircuit of M if and only if every circuit of M that contains one of e and f contains both. **[5]**

Solution: First assume that every circuit that contains one of e and f contains both. We start by showing that $E - \{e, f\}$ is non-spanning. Suppose $E - \{e, f\}$ is spanning. Then it contains a basis B. Then $e \notin B$, and $B \cup e$ contains a unique circuit that contains e. But this circuit does not contain f, so we have a contradiction. Therefore $E - \{e, f\}$ is non-spanning. Hence $E - \{e, f\}$ is contained in a maximal non-spanning set, that is, a hyperplane. Let H be this hyperplane. Now the complement of H is a cocircuit contained in $\{e, f\}$. Since neither $\{e\}$ nor $\{f\}$ is a cocircuit (since these elements are not coloops), it follows that $\{e, f\}$ is a cocircuit.

For the converse, assume that $\{e, f\}$ is a cocircuit, so that $H = E(M) - \{e, f\}$ is a hyperplane. We want to show that no circuit contains precisely one of $\{e, f\}$. Assume, towards a contradiction, that C is a circuit containing (without loss of generality) e but not f. Then $C - e$ is an independent subset of H. Let I be a maximal independent subset of $H \cup e$ such that $C - e \subseteq I$. Observe that $e \notin I$, for otherwise C is contained in the independent set I. Since H is a hyperplane, and e is not in H, it follows that $H \cup e$ contains a basis of M of size $r(M)$. If $|I| < r(M)$, then we could use the axiom **I3** to augment I to a larger subset of $H \cup e$, contradicting our choice of I. Therefore I is a basis of M. But H does not contain a basis of M, so $e \in I$, a contradiction.

Q7. A matroid is *self-dual* if it is isomorphic to its dual. Prove that if M is a self-dual matroid with ground set E, then $|E|$ is even. **[4]**

Solution: Assume that M is isomorphic to M^*. An isomorphism between M and M^* is a bijection such that a set is independent in M if and only if its image is independent in M^*. It follows that a set is a basis in M if and only if its image is a basis in M^*. Therefore M and M^* have the same rank. But then $r(M) = r(M^*) = |E| - r(M)$. Since $r(M) = |E| - r(M)$, this implies $2r(M) = |E|$, so $|E|$ is even, as desired.

Q8. Recall *sparse paving* matroids from Assignment 1. A matroid M is *sparse paving* if and only if every circuit of M has cardinality at least $r(M)$, and whenever C and C' are distinct circuits of M with size $r(M)$, then $|C \cap C'| < r(M) - 1$.[1] **[9]**

(i) Let M be a sparse paving matroid with rank r. Prove that if C is a circuit in M of size r, then C is a hyperplane.

[1] You do not need to prove this statement (it follows easily from the definition seen previously).

Solution: Note that C is not spanning, because it is the same size as a basis, but it is not a basis. In order to show that C is a hyperplane, we must show that it is maximal with respect to being non-spanning. Assume for a contradiction that C is not a hyperplane. Then there is an element, $x \notin C$, such that $C \cup x$ is not spanning. Let y be an element in C. Then $C - y$ is an independent set (because C is minimally dependent), and $|C - y| = r - 1$. If $(C - y) \cup x$ is independent, then it is a basis, because it is the same size as a basis. But in this case, $C \cup x$ is spanning, contrary to assumption. Therefore $(C - y) \cup x$ is dependent, so it contains a circuit. Since $|(C - y) \cup x| = r$, it follows that $(C - y) \cup x$ is actually a circuit itself. But the intersection between C and $(C - y) \cup x$ has cardinality $r - 1$, and this a contradiction. Therefore C is a hyperplane.

(ii) Prove that if M is sparse paving, then M^* is sparse paving.

Solution: Let E be the ground set of M, and let n be $|E|$. Then the rank of M^* is $n - r$. We start by showing that every cocircuit of M has cardinality $n - r$ or $n - r + 1$. Recall that every cocircuit is the complement of a hyperplane. Thus we let H be a hyperplane of M, and let I be a maximal independent subset of H. If x is an arbitrary element not in H, then $H \cup x$ contains a basis. So the rank of $H \cup x$ is r, but the rank of H is less than r. It follows that the rank of H must be $r - 1$. Therefore I is an independent set with cardinality $r - 1$. Firstly, if $H = I$, then the complementary cocircuit has cardinality $n - |H| = n - r + 1$, as desired. Therefore we may assume that there is an element x in $H - I$. Then $I \cup x$ is not independent, so it contains a unique circuit. As $|I \cup x| = r$, it follows that $I \cup x$ must be a circuit. Assume that $y \in H - I$ is distinct from x. Then by the same argument, $I \cup y$ is a circuit of size r. But $I \cup x$ and $I \cup y$ intersect in a set of size $r - 1$, and we have a contradiction. We deduce that x is the only element in $H - I$, so $H = I \cup x$, and the complementary cocircuit has size $n - |H| = n - r$, as desired.

We must also show that if C_1^* and C_2^* are distinct cocircuits with size $n - r$, then $|C_1^* \cap C_2^*| < n - r - 1$. Let H_i be the complementary hyperplane of C_i^*. Then $|H_i| = r$, and $r(H_i) = r - 1$, so H_i must be dependent, and in fact it must be a circuit. Thus $|H_1 \cap H_2| < r - 1$, which means that

$$|C_1^* \cap C_2^*| = |E| - |H_1 \cup H_2| = |E| - (|H_1| + |H_2| - |H_1 \cap H_2|)$$
$$< n - r - r + (r - 1) = n - r - 1$$

as desired.

Q9. Let M be a matroid on the ground set E with r as its rank function. Recall that we use 2^E to denote the power set of E. Define a new function, $r^* \colon 2^E \to \mathbb{Z}$ by the equation $r^*(X) = r(E - X) + |X| - r(M)$ for every subset $X \subseteq E$. Prove directly that r^* satisfies the three conditions **R1**, **R2**, and **R3**, using only the fact that r satisfies **R1**, **R2**, and **R3**, and no other facts about matroid duality. **[8]**

Solution: Let X be any subset of E. Then

$$|X| + r(E - X) \geq r(X) + r(E - X) \geq r(X \cap (E - X)) + r(X \cup (E - X))$$
$$\geq r(\emptyset) + r(E) = 0 + r(M) = r(M).$$

Thus $r^*(X) = r(E - X) + |X| - r(M) \geq 0$.

Also, $E - X \subseteq E$, so $r(E - X) \leq r(E) = r(M)$. Therefore $r(E - X) - r(M) \leq 0$, so $r(E - X) + |X| - r(M) \leq |X|$, which implies that $r^*(X) \leq |X|$. Thus r^* satisfies **R1**.

Assume that $Y \subseteq X$. Then

$$|X| - |Y| + r(E - X) = |X - Y| + r(E - X) \geq r(X - Y) + r(E - X) \geq \\ r(\emptyset) + r((X - Y) \cup (E - X)) = 0 + r(E - Y)$$

This implies $|Y| + r(E - Y) \leq |X| + r(E - X)$, so $|Y| + r(E - Y) - r(M) \leq |X| + r(E - X) - r(M)$. Thus $r^*(Y) \leq r^*(X)$, so r^* satisfies condition **R2**.

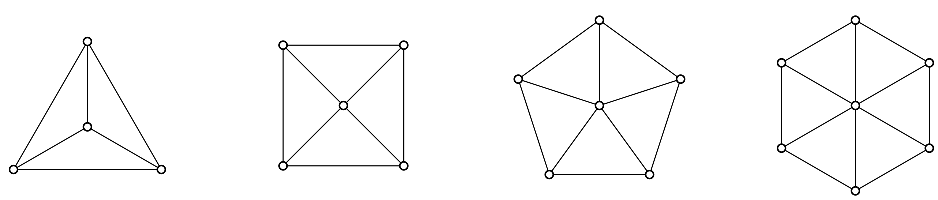
Finally, let X and Y be subsets of E. Then

$$\begin{aligned} r^*(X \cap Y) + r^*(X \cup Y) &= r(E - (X \cap Y)) + |X \cap Y| - r(M) \\ &\quad + r(E - (X \cup Y)) + |X \cup Y| - r(M) \\ &= r((E - X) \cup (E - Y)) + r((E - X) \cap (E - Y)) \\ &\quad + |X \cap Y| + |X| + |Y| - |X \cap Y| - 2r(M) \\ &\leq r(E - X) + r(E - Y) + |X| + |Y| - 2r(M) \\ &= r^*(X) + r^*(Y) \end{aligned}$$

Therefore **R3** holds for r^*.

Q10. Find an infinite sequence of graphs $G_4, G_5, G_6, \ldots$ such that G_i has exactly i vertices for each i, and $M(G_i)$ has a circuit that is also a hyperplane. **[4]**

Solution: The following diagram shows the first few *wheel* graphs. The edges on the outside 'rim' of the wheel form a circuit that is a hyperplane.



G_4 G_5 G_6 G_7