Total marks available: 60

You may use results stated in the course notes, lecture notes, or from previous assignments, but you should cite any results that you use. You should not use any other resources, printed or online. This is not a collaborative assignment – your submitted work should reflect your individual effort.

Q1. Let M be a matroid on the ground set E. Let C be a circuit of M, and let e be an element in $E - C$. Prove that C is a union of circuits in the minor M/e. **[5]**

Q2. Let M be a matroid and let X be a subset of $E(M)$. Prove that the set X is independent if and only if, for every $x \in X$, there exists a flat F_x such that $F_x \cap X = X - x$. **[5]**

Q3. Let $\mathcal{F}$ be a collection of subsets of a set E. Prove that $\mathcal{F}$ is the set of flats of a matroid if and only if the following conditions hold:

(F1) $E \in \mathcal{F}$.

(F2) If $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$.

(F3) If $F \in \mathcal{F}$ and $\{F_1, F_2, \ldots, F_k\}$ is the set of minimal members of $\mathcal{F}$ that properly contain F, then the sets $\{F_1 - F, F_2 - F, \ldots, F_k - F\}$ partition $E - F$.

(You may use any of the axiom schemes from Theorems 6.1–6.4, including those for which we have not seen proofs in lectures.) **[8]**

Q4. Let M be a matroid on the ground set E and let $\{X, Y\}$ be a partition of E into two sets, where X is non-empty. Let x be an element in X. Prove that x is in $\mathrm{cl}^*(Y)$ if and only if x is not in $\mathrm{cl}(X - x)$. **[5]**

Q5. Let $\{e, f, g\}$ be a circuit and a cocircuit of a matroid M. Prove that $M/f\backslash g = M\backslash f/g$.

[6]

Q6. A matroid is said to be *round* if every pair of cocircuits has non-empty intersection. Let M be a round matroid, and let e be an element of $E(M)$. Prove that M/e is round. **[4]**

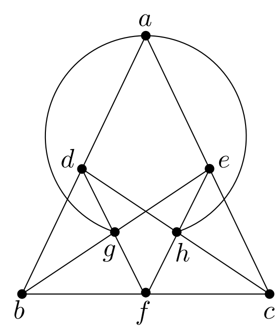
Q7.

(i) Give an example of a matroid M and a weight function $w : E(M) \to \mathbb{R}$ having two distinct bases of maximum weight (thus demonstrating that a solution output by the greedy algorithm need not be unique).

(ii) Let M be a matroid and let $w : E(M) \to \mathbb{R}$ be a injective (one-to-one) function. Prove that M has a unique basis of maximum weight.

[6]

Q8. Let M be the rank-3 matroid shown below.



Assume that $M = M[A]$, where A is the following matrix with entries from some field $\mathbb{F}$.

$$\begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & \zeta & \eta & \theta & \kappa & \lambda \\ 0 & 1 & 0 & \mu & \nu & \xi & \beta & \delta \\ 0 & 0 & 1 & \pi & \rho & \alpha & \gamma & \epsilon \end{bmatrix}$$

Here α, β, γ, δ, ϵ, ζ, η, θ, κ, λ, μ, ν, ξ, π, and ρ are elements of the field $\mathbb{F}$.

(i) Determine which of $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \pi, \rho\}$ are zero, and which are non-zero.

(ii) Give a brief argument that, up to scaling rows and columns of A, we may assume each member of $\{\zeta, \eta, \kappa, \lambda, \mu, \xi, \rho\}$ is 1.

(iii) Prove that $\delta = \gamma = 1$.

(iv) Prove that $\epsilon = \beta^{-1}$.

(v) Prove that $1 - \alpha\beta + \alpha = 0$.

(vi) Prove that $\alpha = \beta^{-1} - 1$.

(vii) Prove that $\beta + \beta^{-1} = 1$.

(viii) Prove that M is not representable over GF(5). **[10]**

Q9. Let M be the rank-4 matroid on ground set $\{1, 2, \ldots, 8\}$ whose set of non-spanning circuits is

$$\{\{1, 3, 7, 8\}, \{1, 2, 6, 8\}, \{2, 3, 4, 5\}, \{2, 3, 6, 7\}, \{4, 5, 6, 7\}\}.$$

(a) Explain why the matroid M is well defined (that is, how do we know that there is a rank-4 matroid on this ground set with this set of non-spanning circuits?).

(b) Draw a geometric representation of M.

(c) Show that this matroid is not representable over $GF(q)$ for $q \in \{2, 3, 4\}$.

(d) By explicitly constructing a $GF(7)$-representation, show that this matroid is representable over $GF(7)$. **[11]**