5 Closure and flats

We start this chapter with two definitions.

Definition 5.1. Let M be a matroid, and let X be a subset of $E(M)$. The *closure* of X, written $\mathrm{cl}(X)$ or $\mathrm{cl}_M(X)$, is the set

$$\{e \in E(M) \colon r(X \cup e) = r(X)\}.$$

Equivalently, $\mathrm{cl}(X) = X \cup \{e \in E(M) - X \colon r(X \cup e) = r(X)\}$. The function cl takes subsets of $E(M)$ to subsets of $E(M)$. We call it the *closure operator* of M. If e is in the closure of X, we say that X *spans* e.

Example. Consider the matroid in Figure 17. The set $\{a, e\}$ spans g, since $\{a, e\}$ and $\{a, e, g\}$ have the same rank. Furthermore, $\{a, e, f\}$ and $\{a, e, h\}$ have the same rank as $\{a, e\}$, so $\{a, e\}$ spans f and h. However, $\{a, e, b\}$ has rank three, which is more than the rank of $\{a, e\}$, so $\{a, e\}$ does not span b. In fact, the closure of $\{a, e\}$ is $\{a, e, f, g, h\}$. ♢

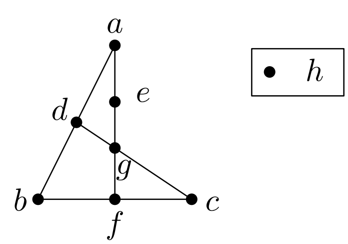

Figure 17: A rank-3 matroid.

Exercise 5.2. Recall that a spanning set is one that contains a basis. For a matroid M and set $X \subseteq E(M)$, prove that X is spanning in M if and only if the closure of X is equal to $E(M)$.

Definition 5.3. A *flat* of a matroid M is a subset $X \subseteq E(M)$ such that $\mathrm{cl}(X) = X$.

In other words, X is a flat if $r(X \cup e) > r(X)$, for every element $e \in E(M) - X$; that is, it is maximal with respect to having rank equal to $r(X)$.

Example. Again consider the rank-3 matroid shown in Figure 17. Recall the convention that, in a geometric representation, loops are shown in a box to one side. The set $\{b, f, c, h\}$ is a flat of this matroid, since if we extend it by adding a new element, its rank will increase from two to three. For the same reason, $\{a, e, f, g, h\}$ and $\{b, g, h\}$ are flats. However, $\{a, b, d\}$ is not a flat, since it has the same rank as $\{a, b, d, h\}$. Similarly, $\{c, d, h\}$ is not a flat, because it has the same rank as $\{c, d, g, h\}$. $\diamondsuit$

Note that X fails to be a flat if and only if there is an element $e \in E(M) - X$ such that $r(X \cup e) = r(X)$. A hyperplane is just a flat with rank $r(M) - 1$.

Exercise 5.4. Prove that if e is a loop of the matroid M, then e is in every flat of M.

Proposition 5.5. *Let E be a finite set, and let r be a function taking subsets of E to integers. Assume that r satisfies the properties* **R2** *and* **R3** *stated on page 9. If X and Y are subsets of E such that $r(X \cup y) = r(X)$ for every element $y \in Y - X$, then $r(X \cup Y) = r(X)$.*

Proof. The proof is by induction on $|Y - X|$. If $|Y - X| = 0$ then Y is contained in X and the result is immediate. Assume that $|Y - X| = t$, where $t > 0$, and that the result holds for all subsets X and Y such that $|Y - X| < t$. Now let y be an element in $Y - X$. Then $r(X \cup y) = r(X)$, by hypothesis, and $r(X \cup (Y - y)) = r(X)$, by induction. Therefore, by applying statements **R2** and **R3**, we see that

$$\begin{aligned} r(X) + r(X) &= r(X \cup (Y - y)) + r(X \cup y) \\ &\geq r((X \cup (Y - y)) \cup (X \cup y)) + r((X \cup (Y - y)) \cap (X \cup y)) \\ &= r(X \cup Y) + r(X) \\ &\geq r(X) + r(X). \end{aligned}$$

(The last inequality follows from applying **R2** to $X \subseteq X \cup Y$.) Since the first and last terms in this sequence of inequalities are equal, all the inequalities must in fact be equalities. Therefore $r(X) + r(X) = r(X \cup Y) + r(X)$. The result follows. $\square$

Note that the rank function of a matroid satisfies **R2** and **R3**, by Theorem 1.16. Therefore we can apply Proposition 5.5 to matroid rank functions.

Proposition 5.6. *Let M be a matroid, and let X be a subset of $E(M)$.*

(i) $r(\mathrm{cl}(X)) = r(X)$.

(ii) $\operatorname{cl}(X)$ *is a flat of* M.

Proof. The first statement follows immediately from the definition of closure and Proposition 5.5. Suppose that the second statement is false. Then there is an element $x \in E(M) - \operatorname{cl}(X)$ such that $r(\operatorname{cl}(X) \cup x) = r(\operatorname{cl}(X))$. Because x is not in $\operatorname{cl}(X)$, it follows that $r(X) \neq r(X \cup x)$. Since **R2** implies that $r(X) \leq r(X \cup x)$, we deduce that $r(X) < r(X \cup x)$. Now $X \cup x \subseteq \operatorname{cl}(X) \cup x$, since $X \subseteq \operatorname{cl}(X)$, so by using **R2** again, we see that $r(X \cup x) \leq r(\operatorname{cl}(X) \cup x) = r(\operatorname{cl}(X))$. Together, these inequalities imply that $r(X) < r(\operatorname{cl}(X))$, which contradicts statement (i). □

In Proposition 5.8, we summarise the properties of the closure operator. First, we require a lemma.

Lemma 5.7. *For a matroid* M *on ground set* E, *suppose* $X \subseteq E$ *and* $e \in E$. *Then*

(i) $r(X \cup x) \in \{r(X), r(X) + 1\}$, *and*

(ii) *if* $x \notin \operatorname{cl}(X)$, *then* $r(X \cup x) = r(X) + 1$.

Proof. Clearly (i) holds when $x \in X$, so suppose $x \in E - X$. By **R2**, $r(X) \leq r(X \cup x)$. By **R3**, $r(X \cup x) + r(\emptyset) \leq r(X) + r(\{x\})$, where $r(\emptyset) = 0$ and $r(\{x\}) \leq 1$, by **R1**. So $r(X \cup x) \leq r(X) + 1$, and hence (i) holds. For (ii), observe that if $x \notin \operatorname{cl}(X)$, then $r(X \cup x) \neq r(X)$, so, by (i), $r(X \cup x) = r(X) + 1$, as required. □

Proposition 5.8. *Let* M *be a matroid with closure operator* cl. *Then* cl *satisfies the following properties:*

CL1. $X \subseteq \operatorname{cl}(X)$ *for all* $X \subseteq E$.

CL2. *If* $Y \subseteq X \subseteq E$, *then* $\operatorname{cl}(Y) \subseteq \operatorname{cl}(X)$.

CL3. $\operatorname{cl}(\operatorname{cl}(X)) = \operatorname{cl}(X)$ *for all* $X \subseteq E$.

CL4. *If* $X \subseteq E$, $x \in E$, *and* $y \in \operatorname{cl}(X \cup x) - \operatorname{cl}(X)$, *then* $x \in \operatorname{cl}(X \cup y)$.

Proof. Since $\operatorname{cl}(X) = X \cup \{e \in E(M) - X : r(X \cup e) = r(X)\}$, **CL1** holds. For **CL2**, let $e \in \operatorname{cl}(Y)$, where $Y \subseteq X \subseteq E$. Then $r(Y \cup e) = r(Y)$. Let B_Y be a basis of Y. Then B_Y is also a basis of $Y \cup e$. Now $X \cup e$ has a basis $B_{X \cup e}$ that contains B_Y. Note that $B_{X \cup e}$ does not contain e, for otherwise $B_{X \cup e} \cap (Y \cup e)$ is an independent set contained in $Y \cup e$ that properly contains

B_Y, contradicting that B_Y is a basis of $Y \cup e$. Since $B_{X\cup e}$ does not contain e, it is also a basis of B_X. So $r(X \cup e) = r(X)$, implying $e \in \text{cl}(X)$. Thus **CL2** holds. Since $\text{cl}(X)$ is a flat, by Proposition 5.6(ii), **CL3** holds.

Finally, consider **CL4**. Suppose $y \in \text{cl}(X \cup x) - \text{cl}(X)$, for some $x \in E$ and $X \subseteq E$. Then $r(X \cup \{x, y\}) = r(X \cup x)$, and $r(X \cup y) = r(X) + 1$, by Lemma 5.7(ii). Now

$$r(X) + 1 = r(X \cup y) \leq r(X \cup \{x, y\}) = r(X \cup x) \leq r(X) + 1,$$

by **R2** and Lemma 5.7(i). So equality holds throughout, implying $r(X \cup \{x, y\}) = r(X \cup y)$. Hence $x \in \text{cl}(X \cup y)$. □

Proposition 5.9. *Let M be a matroid, let X be a subset of $E(M)$, and let $x \in E(M) - X$. Then $x \in \text{cl}(X)$ if and only if there is a circuit C of M such that $x \in C$, and C is contained in $X \cup x$.*

Proof. Suppose that $x \in \text{cl}(X)$. Let I be a maximum-sized independent set contained in X. Then I is also a maximum-sized independent set in $X \cup x$, because $r(X \cup x) = r(X)$. Therefore $I \cup x$ is dependent. Proposition 1.12 implies that there is a circuit C contained in $I \cup x$ that contains x. This proves one direction of the proposition.

For the other direction, assume that such a circuit C exists. Then $C - x$ is an independent set contained in X. Let I be an independent set that is maximum-sized subject to the constraints $C - x \subseteq I \subseteq X$. Suppose that X contains an independent set I' that is larger than I. Then **I3** implies that there is an element $y \in I' - I$ such that $I \cup y$ is independent. But $I \cup y$ contains $C - x$, and is contained in X, contradicting our choice of I. Therefore I is a maximum-sized independent set contained in X, so $r(X) = |I|$.

Now assume that $X \cup x$ contains an independent set that is larger than I. Again, using **I3**, we see that there is an element $y \in (X \cup x) - I$ such that $I \cup y$ is independent. If y is in $X - I$, then I is not a maximum-sized independent set contained in X. Therefore y is not in X, which means that $y = x$. But then, as $C - x \subseteq I$, the set $I \cup y$ contains the circuit C, which contradicts the fact that $I \cup y$ is independent. Hence I is also a maximum-sized independent set in $X \cup x$, so $r(X \cup x) = |I| = r(X)$. This implies that $x \in \text{cl}(X)$, as desired. □

Proposition 5.10. *Suppose that F_1 and F_2 are flats of a matroid M. Then $F_1 \cap F_2$ is also a flat of M.*

Proof. Let E be the ground set of M. Towards a contradiction, suppose that there is an element $x \in E-(F_1 \cap F_2)$ such that $r((F_1 \cap F_2) \cup x) = r(F_1 \cap F_2)$. Then, by Proposition 5.9, there is a circuit C contained in $(F_1 \cap F_2) \cup x$ such that $x \in C$.

Since $C \cup x$ is contained in $F_1 \cup x$, Proposition 5.9 implies that x is in the closure of F_1, so $r(F_1 \cup x) = r(F_1)$. But F_1 is a flat, so this implies that $x \in F_1$. Similarly, $C \cup x$ is contained in $F_2 \cup x$, so $r(F_2 \cup x) = r(F_2)$ by Proposition 5.9, which in turn implies that $x \in F_2$, since F_2 is a flat. Now $x \in F_1 \cap F_2$, contradicting that $x \in E - (F_1 \cap F_2)$. □

Exercise 5.11. Demonstrate that the union of two flats need not be a flat.

Now we know that if F_1 and F_2 are flats, then $F_1 \cap F_2$ is a flat. The union of F_1 and F_2 need not be a flat, but $\mathrm{cl}(F_1 \cup F_2)$ is certainly a flat, and any flat that contains both F_1 and F_2 also contains $\mathrm{cl}(F_1 \cup F_2)$. This shows that the flats of a matroid form a *lattice* under the relation of set inclusion. The *meet* of any two flats is their intersection, and the *join* of any two flats is the closure of their union.

We conclude this chapter by proving some useful facts about circuits, flats, and closure.

Proposition 5.12. *Suppose that C and C^* are, respectively, a circuit and a cocircuit of the matroid M. Then $|C \cap C^*| \neq 1$.*

Proof. Let E be the ground set of M. Assume that C and C^* meet in a single element x. Now $E - C^*$ is a hyperplane of M, by Proposition 3.6. Since x is not contained in the hyperplane $E - C^*$, it follows that $r((E - C^*) \cup x) \neq r(E - C^*)$. But C is a circuit contained in $(E - C^*) \cup x$, and x is contained in C. Therefore x is in the closure of $E - C^*$, by Proposition 5.9, which implies that $r((E - C^*) \cup x) = r(E - C^*)$. Thus we have a contradiction, which proves the result. □

The next result is called the *strong circuit-elimination* axiom, since it is a strengthening of **C3**.

Proposition 5.13. *Let C_1 and C_2 be distinct circuits of a matroid. Let e be an element in $C_1 \cap C_2$, and let f be an element in $C_2 - C_1$. Then there is a circuit C_3 contained in $(C_1 \cup C_2) - e$ such that $f \in C_3$.*

Proof. Let $X = (C_1 \cup C_2) - \{e, f\}$. The circuit C_1 is contained in $X \cup e$, and e is contained in C_1. Proposition 5.9 implies that e is in the closure of

X, so $r(X \cup e) = r(X)$. Now $X \cup \{e, f\}$ contains the circuit C_2, and C_2 contains f. This means that f is in the closure of $X \cup e$, so $r(X \cup \{e, f\}) = r(X \cup e) = r(X)$. But $r(X) \leq r(X \cup f) \leq r(X \cup \{e, f\}) = r(X)$, so we deduce that $r(X \cup f) = r(X)$, and therefore that $f \in \mathrm{cl}(X)$. Proposition 5.9 now implies that there is a circuit contained in $X \cup f = (C_1 \cup C_2) - e$ that contains f, as required. ☐

We can use the closure operator to give a characterisation of the circuits in a single-element contraction.

Proposition 5.14. *Let M be a matroid, let $\mathcal{C}$ be its family of circuits, and let e be a non-loop element in $E(M)$. The family of circuits of M/e is*

$$\{C - e \colon C \in \mathcal{C}, e \in C\} \cup \{C \in \mathcal{C} \colon e \notin \mathrm{cl}(C)\}.$$

Proof. Let C be a circuit of M containing e. Then $C - e$ is dependent in M/e. If X is a proper subset of $C - e$, then $X \cup e$ is a proper subset of C, and is therefore independent in M; therefore X is independent in M/e. This means that $C - e$ is a circuit in M/e. So the family of circuits of M/e contains $\{C - e \colon C \in \mathcal{C}, e \in C\}$.

Next we assume that C is a circuit of M such that $e \notin \mathrm{cl}(C)$. As $C \cup e$ is dependent in M, the set C is dependent in M/e. Let X be any proper subset of C. Then X is independent in M. If $X \cup e$ is dependent, then $X \cup e$ contains a circuit of M, by Proposition 1.12, and this circuit contains e. Then Proposition 5.9 implies that e is in the closure of C, a contradiction. Therefore $X \cup e$ is independent in M, so X is independent in M/e. This shows that C is a circuit of M/e. So the family of circuits of M/e contains $\{C \in \mathcal{C} \colon e \notin \mathrm{cl}(C)\}$.

To complete the proof, we assume C_1 is a circuit of M/e. Then $C_1 \cup e$ is dependent in M, so $C_1 \cup e$ contains a circuit C' of M. Suppose there is some $x \in C_1 - C'$. The set $C_1 - x$ is independent in M/e, so $(C_1 - x) \cup e$ is independent in M. But $(C_1 - x) \cup e$ contains the circuit C', a contradiction. This means that C' contains every element of C_1. If C' also contains e, then $C' = C_1 \cup e$, and C_1 is a member of $\{C - e \colon C \in \mathcal{C}, e \in C\}$. Therefore we suppose that e is not in C', so that $C' = C_1$ is a circuit of M. Towards a contradiction, suppose that $e \in \mathrm{cl}(C_1)$. Proposition 5.9 implies that there M has a circuit C'' contained in $C_1 \cup e$ such that e is in C''. Now C'' cannot be equal to $C_1 \cup e$, for then C'' properly contains the circuit C_1 of M. Therefore $C'' - e$ is properly contained in C_1. But, as C'' is a circuit of M containing e, the set $C'' - e$ is a circuit of M/e, as argued in the first paragraph of this proof. From the contradiction that the circuit $C'' - e$ is properly contained in the circuit C_1 of M/e, we deduce that $e \notin \mathrm{cl}(C_1)$. Therefore C_1 belongs to $\{C \in \mathcal{C} \colon e \notin \mathrm{cl}(C)\}$, as required. □

Finally we introduce a useful matroid operation. A *circuit-hyperplane* is a circuit that is also a hyperplane.

Proposition 5.15. *Let M be a matroid with $\mathcal{I}$ as its family of independent sets, and a circuit-hyperplane X. Then there is a matroid on ground set $E(M)$ whose family of independent sets is $\mathcal{I} \cup \{X\}$.*

Proof. First we note that $r(X) = |X| - 1$, since X is a circuit. Furthermore, $r(X) = r(M) - 1$, as X is a hyperplane. This implies $|X| = r(M)$.

Since $\mathcal{I}$ contains the empty set, so does $\mathcal{I} \cup \{X\}$. Therefore **I1** holds. Every proper subset of X is independent in M, since X is a circuit. Therefore every subset of a member of $\mathcal{I} \cup \{X\}$ is also a member of $\mathcal{I} \cup \{X\}$, and **I2** holds.

To prove **I3**, we assume that I_1 and I_2 are members of $\mathcal{I} \cup \{X\}$ and $|I_2| < |I_1|$. Since $|X| = r(M) \geq |I_1|$, it follows that I_2 is in $\mathcal{I}$. If I_1 is also in $\mathcal{I}$, then $I_2 \cup e$ is in $\mathcal{I}$ for some $e \in I_1 - I_2$, since **I3** holds for $\mathcal{I}$. So we may assume that $I_1 = X$. Suppose $|I_2| < |X| - 1$. Then we can let X' be any subset of X with cardinality $|X| - 1$, in which case X' is in $\mathcal{I}$, since X is a circuit of M. Now we can apply **I3** to I_2 and X', so there exists an element $e \in X' - I_2 \subseteq I_1 - I_2$ such that $I_2 \cup e \in \mathcal{I}$, as required. So we may also assume that $|I_2| = |X| - 1$.

Suppose there is an element $e \in X - \mathrm{cl}(I_2)$. If $I_2 \cup e$ is dependent in M, then Proposition 1.12 implies that $I_2 \cup e$ contains a circuit of M, and e is contained in this circuit. Then Proposition 5.9 implies e is in $\mathrm{cl}(I_2)$, contrary to hypothesis. So $I_2 \cup e$ is in $\mathcal{I}$, and **I3** is again satisfied. Therefore we may also assume that $X \subseteq \mathrm{cl}(I_2)$. Suppose there is an element $e \in \mathrm{cl}(I_2) - X$. Then $X \cup e$ is spanning in M, since X is a hyperplane. But $X \cup e \subseteq \mathrm{cl}(I_2)$,

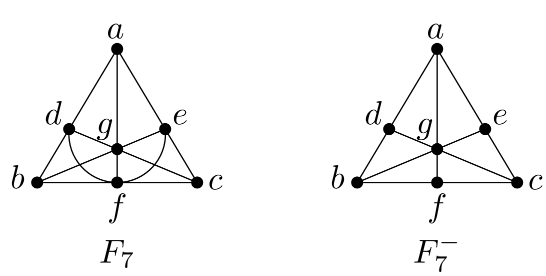

Figure 18: The Fano matroid and the non-Fano matroid.

so $r(M) = r(X \cup e) \leq r(\mathrm{cl}(I_2)) = r(I_2) = |I_2| = |I_1| - 1 = r(M) - 1$, a contradiction. Thus $\mathrm{cl}(I_2) - X = \emptyset$; that is, $X \supseteq \mathrm{cl}(I_2)$. Now $X = \mathrm{cl}(I_2)$. As $I_2 \subseteq \mathrm{cl}(I_2)$, we see that $I_2 \subseteq X$. Recall that $|I_2| = |X| - 1$, so there is a unique element in $X - I_2$; let e be this element, so $X = I_2 \cup e$. Then $I_2 \cup e$ is in $\mathcal{I} \cup \{X\}$, so **I3** is once again satisfied. □

If X is a circuit-hyperplane of M, then we say the matroid with $\mathcal{I}(M) \cup \{X\}$ as its set of independent sets is obtained from M by *relaxing* X. Figure 18 shows the Fano matroid F_7, and the *non-Fano matroid* F_7^-, which is obtained from the Fano matroid by relaxing the circuit-hyperplane $\{d, e, f\}$.